THE

# MATHEMATICAL GAZETTE

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# SOME EQUATIONS CONNECTED WITH THE PLANE SECTION OF THE CONICOID.

By H. G. Green

We propose in this paper to develop equations which give the principal points and lines of the conic formed by the intersection of the general conicoid.

$$S=ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0,$$

with the plane $lx+my+nz+p=0$, in a form convenient for numerical work. We discuss the general case in which special values of the constants do not lead to special cases: the slight modifications necessary where such occur become clearly manifest in numerical work. For brevity, the expressions $ax+hy+gz+u$, etc., will be denoted by the usual symbols $X$, $Y$, $Z$, and for $nY-mZ$, $lZ-nX$, $mX-lY$ we will write $U$, $V$, $W$ where $lU+mV+nW\equiv 0$. We can assume without loss of generality that $n$ is not zero.

1. *The foci.*

Let $x'$, $y'$, $z'$ be a focus of the section made by the plane. The tangent cone from the focus to the conicoid has four generators which intersect the circle at infinity, and two of these generators must lie in the plane of section. Referred to the focus as origin, all cones which have these four generators as generators are included in the equation

$$S'(ax^2+by^2+cz^2+2fyz+2gzx+2hxy) - (xX'+yY'+zZ')^2-\lambda(x^2+y^2+z^2)=0, \quad ...(1)$$

and for some value (or values) of $\lambda$ this must reduce to two planes of which $lx+my+nz=0$ is one. Substituting $-(lx+my)/n$ for $z$ in (1), supposing $n$ not zero, we obtain a quadratic in $x/y$ whose coefficients for a suitable value of $\lambda$ must vanish identically, or for this $\lambda$ the coordinates $x'$, $y'$, $z'$ of the focus must satisfy the equations

$$S(an^2+cl^2-2gln)-V^2-\lambda(n^2+l^2)=0, \quad .....(2)$$

$$S(bn^2+cm^2-2fmn)-U^2-\lambda(n^2+m^2)=0, \quad .....(3)$$

$$S(clm-fnl-gnm+hn^2)+UV-\lambda lm=0, \quad .....(4)$$

where $n$ is not zero. One other equation of type (2), (3) and two more of type (4) can be obtained in a similar manner with manifest modifications when $l$ or $m$ is zero. From three of these equations, by elimination of $\lambda$, we can obtain the equations of two surfaces whose line of intersection gives the locus of the foci of plane sections parallel to $lx+my+nz=0$.

2. *The axes.*

From equations (2), (3), (4) we have as the equation of one surface containing the foci

$$\begin{vmatrix} an^2+cl^2-2gln & V^2 & l^2+n^2 \\ bn^2+cm^2-2fmn & U^2 & m^2+n^2 \\ cln-fnl-gnm+hn^2 & -UV & lm \end{vmatrix}=0, \quad \ldots\ldots(5)$$

but this, in general, is the equation of two planes through the line of intersection of $U=0$, $V=0$ (*i.e.* through $X/l=Y/m=Z/n$) and hence, with the equation of the plane section, gives the axes. The equation can be written in the symmetrical form

$$\begin{vmatrix} amn-glm-hln+fl^2 & VW & mn \\ bnl-hmn-fml+gm^2 & WU & nl \\ clm-fnl-gnm+hn^2 & UV & lm \end{vmatrix}=0, \quad \ldots\ldots\ldots\ldots(6)$$

which may be obtained by reduction from (5) or immediately from the equations of type (4). The form (5) is, however, the more convenient for numerical work.

3. *The auxiliary circles.*

If $L=0$ is one of the axial planes derived from (5), the conic formed by the intersection of the plane of section with the surface $S-\mu L^2=0$ has double contact with the conic section, with an axis of the latter as chord of contact. If, therefore, we adjust $\mu$ so that the plane of section is a plane of circular section of $S-\mu L^2=0$, the surface thus obtained will contain the auxiliary circle required. The coefficients of second degree only enter into the two conditions for circular section which, from the choice of $L$, are simultaneously satisfied by the same $\mu$.

(*Note.*—The demonstrations, as usually given in the text-books, for the conditions of circular section are applicable only to central conicoids. The results, however, are quite general.)

4. *The director circle.*

Transferring the origin to a point $x'$, $y'$, $z'$ in the plane of section, the tangents from it are given by the equations

$$S'(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)-(xX'+yY'+zZ')^2=0,$$
$$lx+my+nz=0.$$

The condition for perpendicularity is immediately

$$\Sigma\{(S'b-Y'^2)n^2-2(S'f-Y'Z')nm+(S'c-Z'^2)m^2\}=0,$$

or, rearranging and dropping the dashes, the equation of the director circle is given by

$$\Sigma\{S(bn^2+cm^2-2fnm)-U^2\}=0, \quad \ldots\ldots\ldots\ldots\ldots(7)$$

with the equation of the plane of section.

5. *The directrices.*

From equation (7) the conics passing through the points of intersection of the plane section and its directrices are given by

$$\lambda S-(U^2+V^2+W^2)=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

with the equation of the plane. For three values of $\lambda$ this will give two straight lines: since $lU+mV+nW \equiv 0$, one of these values will be zero, and we obtain, as is easily seen from the generalised figure, the joins of the centre to the circular points—the two others give rise to the pairs of directrices.

It is a necessary condition that the terms of second degree in (8) should factorise—that it is sufficient follows from the form of (8) and its relationship to the plane. The discriminating cubic of these terms, therefore, gives the required values in $\lambda$ together with a zero root which we discard.

6. *The asymptotes.*

If $x', y', z'$ be a point on an asymptote its polar line in the section must be parallel to a generator of the tangent cone from $x', y', z'$ to the conicoid, or taking the parallels through the origin the line $lx+my+nz=0$, $xX'+yY'+zZ'=0$ must be a generator of

$$S'(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)-(xX'+yY'+zZ')^2=0.$$

The line may be written $x/U'=y/V'=z/W'$, and for the asymptotes we have (with the equation of the plane)

$$\Sigma aU^2+2\Sigma fVW=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

This surface (which from the linear identity between $U$, $V$, $W$ consists of two planes) contains in particular the generators of the conicoid lying in the tangent planes parallel to the plane of section.

7. *Numerical example.*

Take the section of $4x^2+y^2-4z^2-12=0$, by the plane

$$x+y+z+3=0.$$

We have

$$U=Y-Z=y+4z, \quad V=Z-X=-4z-4x, \quad W=X-Y=4x-y.$$

Equation (5) for the axes becomes

$$\begin{vmatrix} 0 & V^2 & 2 \\ -3 & U^2 & 2 \\ -4 & -UV & 1 \end{vmatrix}=0,$$

or

$$5V^2-6UV-8U^2=0,$$

and the axes lie on the planes

$$V-2U=0, \quad 5V+4U=0, \quad \text{or} \quad 2x+y+6z=0, \quad 5x-y+z=0.$$

For the foci, from equations (2), (3),

$$S(0) - V^2 - 2\lambda = 0, \quad S(-3) - U^2 - 2\lambda = 0.$$

Eliminating $\lambda$, the foci lie on

$$3(4x^2+y^2-4z^2-12) - 16(x+z)^2 + (y+4z)^2 = 0. \quad \ldots\ldots\ldots(a)$$

The axis $x+y+z+3=0$, $2x+y+6z=0$ can be written

$$x = -5z+3, \quad y = 4z-6,$$

and substitution in $(a)$ leads to $2z^2-3z+1=0$, or the real foci are

$$\{\tfrac{1}{2}, -4, \tfrac{1}{2}\} \text{ and } \{-2, -2, 1\}.$$

For the auxiliary circle on the major axis we take $\mu$, such that

$$x+y+z+3=0$$

is a circular section of

$$4x^2+y^2-4z^2-12-\mu(2x+y+6z)^2=0.$$

We have therefore

$$\frac{(1-\mu)+(-4-36\mu)+12\mu}{2} = \frac{(-4-36\mu)+(4-4\mu)+24\mu}{2}$$

$$= \frac{(4-4\mu)+(1-\mu)+4\mu}{2}.$$

These are satisfied when $\mu = -\frac{1}{2}$, and the auxiliary circle is the intersection of $x+y+z+3=0$, with

$$3(4x^2+y^2-4z^2-12)+(2x+y+6z)^2=0.$$

Equation (7), which, with that of the plane, gives the director circle, becomes

$$(4x^2+y^2-4z^2-12)(-3+0+5) - (y+4z)^2$$
$$-(-4z-4x)^2-(4x-y)^2=0,$$

or
$$3x^2+5z^2+yz+4zx-xy+3=0.$$

For the directrices, we have from (8) the equation in $\lambda$,

$$\begin{vmatrix} 4\lambda-32 & 4 & -16 \\ 4 & \lambda-2 & -4 \\ -16 & -4 & -4\lambda-32 \end{vmatrix} = 0,$$

with roots 0 (which we discard), 8 and $-6$. On taking $\lambda = -6$ equation (8) reduces at once to

$$28x^2+4y^2+4z^2+4yz+16zx-4xy-36=0,$$

or $\{(2y-x+z)+\sqrt{-3}(z+3x)\}\{(2y-x+z)-\sqrt{-3}(z+3x)\}-36=0.$

On the plane of section, $y=-(x+z+3)$, and the equation becomes

$$\{(z+3x+6)-\sqrt{-3}(z+3x)\}\{(z+3x+6)+\sqrt{-3}(z+3x)\}-36=0,$$

or
$$(z+3x+6)^2+3(z+3x)^2-36=0,$$

or
$$(z+3x)(z+3x+3)=0.$$

The real directrices therefore lie in the planes

$$z+3x=0,\ z+3x+3=0,$$

and by virtue of the equation of the plane of intersection,

$$x+y+z+3=0,$$

they may conveniently be given as lying in the planes $5x-y+z=\pm 3$ (cf. the equation of the plane giving the minor axis).

For the asymptotes, equation (9) gives

$$4(y+4z)^2+16(z+x)^2-4(4x-y)^2=0,$$

or
$$(z+x)^2+(z+x)(4z+2y-4x)=0.$$

The asymptotes, therefore, lie on the planes

$$x+z=0,\ 3x-2y-5z=0.$$

(*Note.*—Although we give numerical illustration of the application of all the general formulae, it is often easier in a special case to obtain equivalents to some of them by indirect methods. For example, if the centre of the section be found, a point on an axis may be defined by the property that its polar (the intersection of the polar plane with the plane of section) is perpendicular to its join to the centre, and the asymptotes may be found as the lines of intersection of the tangent cone from the centre with the plane of section; equally well, if the foci are known, the directrices may be obtained from their polar planes. It will be noted, however, that much of the work in handling the general equations will be common to all members of a series of parallel planes.) H. G. G.

## GLEANINGS FAR AND NEAR.

**916.** They (the Jesuits) were admirable teachers of mathematics.—J. Morley, *Miscellanies*, ii. p. 167. [Per Mr. A. Buxton.]

**917.** It is true that a geometer, too, works for the public weal.—J. Morley, *Miscellanies*, ii. p. 169. [Per Mr. A. Buxton.]

**918.** Mathematics has always been in the eyes of the Church a harmless branch of knowledge.—J. Morley, *Miscellanies*, ii. p. 168. [Per Mr. A. Buxton.]

**919.** My parents paid a governess to teach me to read and write and do a few simple sums in arithmetic. The arithmetic did not get beyond simple addition and subtraction, because the poor old lady was incapable of explaining the nature of division.—George Bernard Shaw, in a broadcast talk. [Per Rev. E. M. Radford.]

# CONTINUITY AND IRRATIONAL NUMBER.

By G. J. Whitrow.

In a previous article * I have outlined my conception of the rôle of history in the exposition of mathematical technique. In this article I attempt to provide my bare thesis with a respectable clothing of practicability. The treatment of Number and Continuity which follows is a short and very sketchy historical supplement to the technical treatment of the same subjects in, say, the earlier chapters of Hardy's *Pure Mathematics*.

Primitive man's conception of the world was based on his perception of universal difference and dissimilarity. In the prelogical stage "space" did not appear as homogeneous—in fact to this day the less developed races still retain the faculty of "orientation". Without the idea of the homogeneity of space, however, a science of geometry was impossible. Thus the preliminary step to the construction of such a science was the creation of a fictitious world. Moreover, between any two distinct shapes in visual space, despite every quantitative similarity, there is at least a qualitative difference in so far as they make separate contributions to the pattern presented to us. It was necessary, therefore, that the geometer should construct a space which was not the image of the space of sight, but "the projection and continuation of that impulse to spatiality already visible in visual perception to its ideal limit". Similarly an act of fiction was necessary to the development of arithmetic. Cardinal number and tallying arose from "equating the unequal" and assuming that there is a certain homogeneous element common to all discrete objects—a numerical element. In order to estimate and to give a name to any cardinal number it was ultimately necessary, historically if not logically, to develop the idea of ordinal number, and this involved an analysis of the term "many". Thus, at the very threshold of mathematics, man was faced with the fundamental polarity of synthesised space and analysed number, and so, in a sense, of continuity and discontinuity.

We must look to the Greeks for the earliest rational speculations on number and continuity. It is important to realise how keen was the Greek insight into the order and harmony of nature. As a French writer has remarked: "Le Grec aperçoit l'ordre dans les choses, le moderne l'y introduit". For the early Greek philosophers space and number were essentially *material* concepts. Thales' "water", Anaximenes' "air", Heraclitus' "fire" were anticipations of the "infinite"—by which was implied the notion of cosmic matter diffused throughout and identical with space. Pythagoras conceived the idea of the condensation of the primitive infinite around nomads as centres so that points became delimited. This idea may have been suggested to him as a result of his astronomical studies, for the framework of the heavens appeared to be an

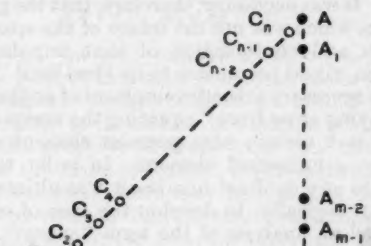
* "The Importance of the History of Mathematics in relation to the Study of Mathematical Technique", *Math. Gazette*, XVI (October 1932), p. 225.

aggregate of points. Thus we see that for Pythagoras the point was "concrete", it was "unity having position" and was *not* dimensionless.

The central doctrine of Pythagoras' philosophy was his assertion that "things are (like) numbers". Much useless verbiage has resulted from attempts to analyse this semi-mystical declaration, and it is well to bear the following points in mind:

(1) that a corpuscular theory of matter was implied, and this involves the existence of indivisibles or "spatial units";
(2) that by a "number" Pythagoras implied a "concrete number", *i.e.* an assemblage of points in definite order and formation, and thus
(3) that in less epigrammatic language the dictum becomes, "All things are composed of spatial units which taken together constitute a number".

It has taken long ages for "number" to be deprived of its many non-mathematical associations. Leaving the theological and kabbalistic aspects out of account, one is still left to distinguish between logical and organic unity. Pythagoras' number includes both. Moreover, as will be seen, it includes measurable as well as numerical unity.




Apart from theoretical arguments, Pythagoras had at least one very important piece of experimental evidence for his doctrine. By employing a monochord, consisting of a string stretched over a resounding board with a movable bridge, he was able to measure the lengths corresponding to high and low notes on the same string. The significance of this experiment lies in the fact that it demonstrated a correlation between numbers and sound (one of the most volatile of phenomena). Nevertheless this deduction was based on the confusion of ratios of lengths with ratios of numbers. The importance of this confusion cannot be overemphasised. Unit length is a magnitude qualitatively indistinguishable from other magnitudes of the same kind, and if we identify it with numerical unity we are led either to treat the number one as divisible or to admit an

indivisible length. Either of these eventually would involve a series of antinomies.

At this stage we ought to remind ourselves that fractions were never regarded by the Greeks as numbers in themselves, but as ratios of integers. Pythagoras' doctrine of proportion was based on his corpuscular theory of matter—for him there existed an indivisible length, "the point", and any two lines stood to one another in a definite numerical ratio. For a time geometry became a branch of arithmetic, until the implications of Pythagoras' theorem came to light, and it was discovered that the diagonal of a square is "incommensurable" with respect to the sides. There was only one logical escape from this dilemma, unless one rejected the entire fabric of Pythagoras' system, and that was to assert that the diagonal of a square cannot be constructed.*

This difficulty was the result of the conflict of continuity and discontinuity—of homogeneous space and discrete number. It is important to notice that the Pythagoreans explicitly recognised the gaps in the number system. The logical consequence, therefore, should have been to recognise gaps in lengths "between the points". But they never seem to have faced this "fact", and it was reserved for Parmanides and his pupil Zeno to examine it with scientific ruthlessness. Zeno's so-called sophisms or paradoxes were simply *reductiones ad absurdum* of the Pythagorean thesis. They reveal that the existence of gaps in space necessarily implies the existence of gaps in time. Strictly speaking, Zeno was not talking the language of geometry but of kinematics, so to the Greek geometers his paradoxes may have appeared irrelevant. To modern mathematicians, however, the particularly relevant aspect is to be found in the $\epsilon$ argument. Most of modern analysis is ultimately based on inequalities of the form $x < \epsilon$, where $\epsilon$ is arbitrarily small. In Zeno's paradoxes the fundamental inequality is $x > \epsilon$, where $\epsilon$ is a (theoretically) fixed lower limit. Now for the Greek, I think it was an integral part of his outlook that there should be a fixed lower limit, $\epsilon$. I think that to him the source of the dilemma in these paradoxes appeared to be the law of ordinal arithmetic, viz. that every integral number has a successor, for was it not this which made possible the existence of an infinite sequence? In him it was not "horror of the infinitesimal and indivisible" but "horror of the infinite process" which the Eleatic arguments inspired. Arithmetic, *quâ* arithmetic, was definitely brought into disrepute.

In this rapid survey of the Greek attitude towards our subject there are two other names which must not be omitted. Archimedes is famous for many things, but in this context he is chiefly famous for his brilliant use of the "method of exhaustions", originally devised by Eudoxus, one of the greatest pure mathematicians the

* My diagram illustrates what this sentence means. $A$ comes between $C_n$ and $C_{n+1}$ on $CC_1C_2\ldots$ and thus the constructed diagonal "falls short of" or "goes beyond" $A$.

*N.B.*—To the Pythagoreans the "strips" $C_1C_2$, etc., and *not* the partitions $C_1$, $C_2\ldots$, etc., were the "points" of the diagonal.

world has ever seen.* This method was the Greek equivalent to our Newtonian Integral Calculus, and depended on the notion of upper and lower approximations to the required answer. It was based on the important axiom (due to Eudoxus)—now known as Archimedes' axiom: "however small $x$ may be and however big $y$ there exists an integer $n$ such that $nx > y$". One notices that $x$ may never be zero nor $y$ infinite, or, more precisely, any number system to which this axiom applies must include or exclude infinity and zero together. As we know, the Greeks excluded both, and it was reserved for the Hindus and Arabians to pave the way for the modern "solution" of the problem of correlating the spatial and the numerical.

During the sixteen or more centuries which elapsed between the death of Archimedes and the rise of the Renaissance mathematicians, the concept "number" was gradually extended. Symbolism was the primary cause. The idea of indeterminate number, revealed par excellence in Algebra, compelled mathematicians to resort to symbolic solutions, irrespective of whether they had a meaning or not. But to write down the meaningless is to endow it with some meaning, and so gradually zero, fractions, and later negative numbers were admitted on equal terms with integers. It was not unnatural that such an extension of the idea of number should have been slow and painful, and it is not surprising to discover that unity was not regarded as a number until a very late date. The Greeks regarded it as the origin of numbers. For them a number was essentially a plurality; unity was therefore never conceived as a number in the full meaning of the word. This idea survived for a long time, and we find in the Arithmetic (1585) of no less celebrated a mathematician than Stevinus of Bruges a long logical discussion establishing one as a number. Despite his logical pedantry on this point he was not squeamish about treating irrationals as if they obeyed the same laws as ratios. The time was approaching, in fact, for a new coalescence of geometry and arithmetic.

In the early seventeenth century Descartes invented his Analytical Geometry on the assumption that anything which is to unity as one linear segment to another may be regarded as a number. It is important that we should be clear on the following:

(1) that in the winter of 1619-20 Descartes studied the mathematics of Pythagoras and thus was fully aware of that system;

(2) that the Pythagorean and Cartesian number concepts were entirely different. Pythagoras correlated *lengths* and *numbers*. Descartes (though he used the language of mensuration) implicitly, at least, correlated *points* and *number-sets*, or coordinates. Just as unity may be regarded numerically and metrically, so zero may be regarded on the one hand as the symbol for the null-class, and on the other as the origin of a frame of reference. It was the latter idea which Descartes

* Eudoxus and Archimedes are, in a sense, parallel to Newton and Gauss, respectively.

introduced and as a result of which he was able to attach a specific numerical label to every point in virtue of its position with respect to zero and the axes. Despite superficial similarity, the analytical geometries of Descartes and of Apollonius were fundamentally different, and that of Descartes was by far the more fruitful. As a result of this, guided by intuition, and unperturbed by logic, geometers saw their ancient science develop an amazing second youth.

Meanwhile, with just as little respect for logic, the problem of infinitesimals was treated by Cavalieri, Kepler, Fermat, Barrow and eventually by Newton and Leibnitz. In the Differential Calculus, as in Analytical Geometry, the practical results were so astounding that these alone seemed more than sufficient justification for the validity of the theory. Nevertheless, the logical difficulties were still present. Newton spoke of a flow and of fluxions but he determined these fluxions by treating the flow as if it were a succession of minute jumps. A Zeno, however, was destined to arise to point out this and many other analogous antinomies. Bishop Berkeley smote the mathematicians hip and thigh, pointing out with much logic and no less wit that infinitesimals had to fulfil simultaneously the contradictory properties of being both equal to zero and different from zero. As a result of his polemics the subsequent treatment of the calculus in England, where it was regarded as a branch of geometry, was, from the standpoint of rigour, far in advance of the continental, though achieving far fewer spectacular results.

On the continent, however, logic was almost abandoned. As Spengler ironically remarks, "The age of refined scepticism saw the emergence of one seemingly impossible truth after another". Thus, for example, infinite series were used irrespective of whether they were convergent or divergent, and infinite processes such as integration and summation were inverted purely at the whim of the writer and independent of any logical criteria. With the turn of the century logic began slowly to emerge out of the welter of new tricks and devices with which the mathematical market had been flooded. Cauchy paved the way, but the fundamental problem, with which we are concerned here, was not really tackled until the middle of the nineteenth century by Dedekind, Weierstrass and Cantor. Meanwhile the number domain had undergone more extensions. Abel's work on the quintic in 1826 revealed the "existence" of algebraic numbers not expressible either as rationals or as radicals, and in 1844 Liouville discovered the class of transcendentals, to which $e$ and $\pi$ were later shown to belong.

Cauchy defined an irrational number as a limit. By a limit he understood a certain fixed value towards which a variable approaches indefinitely near. If this limit is not a rational number, then if we start from the field of rational numbers the limit is fictitious, and thus if we define irrationals as limits they become logically non-existent in our arithmetic. The first person to evade this difficulty successfully and to divorce the apparently incompatible notions of irrationals and limits was Dedekind—in his famous pamphlet,

*Stetigkeit und irrationale Zahlen* (published 1872). The essential idea occurred to him in 1858, but it was dependent, implicitly at least, on the "principles of permanence" first stated by Hankel in 1867. These may be stated, as Danzig (*Number: the language of Science*) states them, in the form of the following definition:

"A collection of symbols, infinite in number, is called a number field, and each individual element in it has a number

(1) if among the elements of the collection we can identify the sequence of *natural numbers*;

(2) if we can establish criteria of rank which will permit us to tell of any two elements whether they are equal or, if not equal, which is the greater, these criteria reducing to the natural criteria, when the two elements are natural numbers;

(3) if for any two elements of the collection we can devise a scheme of addition and multiplication which will have the commutative, associative and distributive properties of the characteristic operations bearing these names and which will reduce to the natural operations when the two elements are natural numbers".

Summarising all this, we may say that if any system submits to operations *as if* it were composed of bona-fide numbers then from the mathematical point of view we may actually regard it *as* a system of numbers. With this idea latent in his mind Dedekind framed the axiom now known after him and Cantor conjointly: "It is possible to assign to any point on a line a unique *real* number, and conversely any real number can be represented in a unique manner by a point on a line". This axiom, like so many axioms, was simply a definition —the arithmetical definition of a straight line. In essence it is fundamental in Cartesian geometry.

To my mind the striking thing about Dedekind's theory is his conception of a point. I think that for him, as for Clifford, a point "is not an idea got at by assuming a small particle to become smaller and smaller without any limit, but it is the boundary between two adjacent portions of a line, which is the boundary between two adjacent portions of a surface, which is the boundary between two adjacent portions of space". The classical notions of a point either as the generator of a line or as the limit of a decreasing magnitude had involved a host of antinomies. By introducing the notion of a point as a partition of a line, and correlatively of an irrational as the partition of rational numbers, Dedekind appeared to have abolished completely the infinitesimal and the infinite process. Actually this was not the case, for, as we can easily see, the partition process becomes trivial if applied to only a finite set of rationals. It is essential that we should partition *all* the rational numbers. We seem inevitably driven to the conclusion that in order to connect the continuous and the discontinuous, geometry and arithmetic, we must invoke the infinite.

Cantor's theory, of which Weierstrass's is a particular case, proceeds entirely differently. Just as Frege and Russell define the cardinal number 2, say, as the class of all couples, so Cantor regarded

the irrational as defined by, and therefore equivalent to, an infinite convergent sequence of certain specified rationals. Thus an irrational was really not one number but a specified infinite set of rational numbers. Moreover, convergence was defined in a purely arithmetic manner, making no reference to infinitesimals.

In conclusion, let us examine the Cantor-Dedekind notion of the linear continuum, on which so much modern analysis is based. Any real number is represented by a point on a line. The totality of real numbers obeys the following rules :

(1) it is well ordered ;
(2) it is infinite in extent ;
(3) it is everywhere dense (between any two distinct numbers a third may be constructed) ;
(4) it contains all its limit points ;
(5) it contains as a sub-domain the totality of rational numbers.

This sub-domain obeys (1), (2) and (3), but not (4). It is therefore said to be " compact ", whereas the real domain is said to be " perfect ", and is thus said to be a " continuum ". This notion of the continuum is a highly ingenious theoretical substitution for the vague empirical notion of the continuous. Indeed, from the point of view of the physicist and applied mathematician, no guess could have been more wild and further from the truth than Pythagoras' cry, " All things are numbers ". In any but a highly academic sense, irrationals are *not* numbers. As Sir Oliver Lodge in his British Association address on " Continuity " (1913) pointed out, " Whenever a commensurable number is really associated with a natural phenomenon there is necessarily a noteworthy circumstance involved in the fact ". Bearing this in mind it is not so paradoxical as Danzig (*vide Number : the language of Science*) seems to imagine that since the present acts as a partition and is neither part of the past nor part of the future it is truly irrational in the Dedekind sense.

G. J. WHITROW.

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**920.** Warren, picking up a stone flung it out far. It described its semicircle, and sank into oblivion again, in the wilderness of thorns and scrub and other stones.—Edward Thompson, *Lament for Adonis*, p. 173. [Per Mr. R. C. Fawdry.]

**921.** Some faint smattering of Arithmetic, or the everlasting laws of Numbers ; faint smattering of Geometry, everlasting laws of Shapes ; these things, we guess, not altogether in the dark, Rodriguez Francia did learn, and found extremely remarkable. Curious enough : that round Globe put into that round Drum, to touch it at the ends and all round, it is precisely as if you clapt 2 into the inside of 3, not a jot more, not a jot less : wonder at it, O Francia ; for in fact it is a thing to make one pause ! Old Greek Archimedeses, Pythagorases, dusky Indians, old nearly as the hills, detected such things ; and they have got across into Paraguay into this brain of thine, thou happy Francia. How is it too, that the Almighty Maker's Planets run, in those heavenly spaces, in paths which are conceivable in thy poor human head as Sections of a Cone ? The thing thou conceivest as an Ellipsis, the Almighty Maker has set his Planets to roll in that.—T. Carlyle, *Dr. Francia : Miscellanies* (1888), iv, p. 266.

## THE STUDY OF STATISTICS IN A SCHOOL COURSE.*

**Mr. F. Sandon** (Plymouth): I have been asked to open a discussion on the subject of the study of statistics in a school course, and I have only accepted the invitation of the Programme Committee with great diffidence. In the first place I am only too conscious of my limitations to speak on the subject, being neither professional statistician nor greatly experienced in teaching the subject in schools. In the second place I am not convinced that there is any place for statistics as such in the school course. When I put these points to Miss Griffiths she found in my very reasons an answer to my plea to be excused this paper. So I am here to think aloud why I have come to my second conclusion.

To begin with, we must go down to the purpose of a good education. We have been told that it should produce the faculty of criticism, discrimination and taste, or, by a speaker at our last Annual Meeting, that the last school years should be spent in learning how to read, how to write and how to think. The last criticism is in my judgment too purely intellectual, the former too vague. I should like our pupils to leave school with an impartial, international, generous, disciplined and restrained mind capable of appreciating beauty and testing conventional judgments with intelligence and taste. But if I try to apply this and say that this can be done by teaching biology, Esperanto, architecture and economics, I find it impossible to fit these into a school time-table, even if I jettison history and literature as taught to-day as being either meaningless facts or someone else's secondhand opinions that only a precocious child with unexampled insight could possibly in the absence of its life's experience reach and understand intelligently on its own initiative. And with the subjects referred to I class statistics. An increasing field of investigation lies open to the statistician, and his calculus is being used more and more in all sorts of problems. And more and more will it be possible to prove anything by statistics to the uncritical, untrained, or indolently minded. But with statistics as with the other subjects I feel that it is to be rather our object to indicate that there are methods and problems and studies and leave them to be acquired and pursued by the intellectually awakened. Our job is to stimulate this intellectual awakening. Well, how do we do it? There are three factors concerned:

(i) the pupil;
(ii) the teacher;
(iii) the subject matter.

With regard to the first, the statistician's point of view has in my opinion had far too little emphasis given to it. Every teacher quickly realises that children are not equal in capacity, and that despite imperfect correlations between subjects, on the whole and in the long run the brighter at one subject is brighter at another.

* A discussion at the Annual Meeting, 6th January, 1933.

The effect of selection for schools of special type is to reduce this correlation somewhat but it is observable in the most homogeneous set. Psychologists and others with intelligence tests, standardised examination papers and statistical analyses applied to school and examination marks have given us some measure of the scatter in intellectual ability. It is amazingly large. From the present point of view it is too great for the majority of the population, in my judgment, to grasp many of the more elementary ideas of the statistical calculus. An alarming prospect for future schoolmasters and statesmen and one for the eugenists to speak about. All we can hope to do with the average child, that is, one of Intelligence Quotient 100 (and Thorndike, I think, says somewhere that simple algebraic ideas are beyond the grasp of children of I.Q. less than 110), is to indicate the very simplest notions of statistics.

With regard to the second factor, the teacher, few teachers even of mathematics have in general even the vaguest ideas of statistics. A school organisation should make the sets of pupils as equal in capacity as possible and then work each set at such a pace that the differences between the individuals are as marked as possible. For the standard of the set there should be some excellent pupils and some very bad—though relative to other less selected groups the pupils may all be mediocre. But teachers, even mathematics teachers, still worship the idea of some absolute standard—of marks, of capacity, of what not. Until teachers have some notion of statistical ideas we cannot hope for much development. And their statistics they must grasp for themselves, for, to judge by the publications of members of staffs of training colleges and the like, the comprehension of statistical notions and the background these supply is very weak. Teachers will have to get their statistics for themselves by private reading.

Bearing in mind these observations relative to the pupil and to the teacher, what is to be said about the subject matter? I think we can introduce modern statistical ideas into our elementary work in one or two places quite conveniently, usefully and profitably.

(1) In arithmetic averages.

The ordinary airthmetic text-book invents a miscellaneous collection of measures of one particular unit, finds the sum of them and divides by the number of items and calls that the average. This is unsatisfactory. An average should represent a general tendency, and the average age of my grandfather, my father, myself and my son does not represent anything usefully. The average age of a form does represent something: it represents the norm, a measure for that class by which we can compare it with the class higher or the class lower. So we ought to emphasise this idea of a tendency in dealing with averages and take as examples groups possessing a common tendency—for example, ages of individuals in a form, hours of sunshine of days in a month, scores per innings in cricket, receipts per week of the local tramway undertaking. With this we can point out the simple rule of using a working mean and taking positive and negative deviations from it, as Durell and Fawdry do on p. 36 of their

*Arithmetic.* But we can point out other measures of the tendency—the median, or the measure of the halfway individual, and the mode, or the commonest measure. Other associated ideas such as G.M.'s and measures of scatter (quartiles, deviations, etc.) are not usually appropriate. Some pupils, however, can appreciate here, or later, weighted means, with something about Ministry of Labour retail prices, index numbers, and a warning of a fallacy exemplified in the following cases.

The first example is taken from a paper published by the Medical Research Council in 1922, by Dr. Brownlee, on "The Use of Death Rates as a Measure of Hygienic Conditions".

| Age. | 45- | 55- | 65- | All, 45 and upwards. |
|---|---|---|---|---|
| Death rate of Clergy - | 10 | 23 | 83 | 35 |
| Death rate of Railway Engine Drivers - | 13 | 30 | 93 | 27 |

These are the death rates of clergymen, or people described on the death certificates as clergy, and of people described on the death certificates as railwaymen. These death rates are at ages 45, 46 and so on, up to but not including 55, etc. The death rates are the rates per thousand of that type of population living, as estimated by the Census. You notice that if you simply take the final figure which is obtained by taking the total number of clergymen aged 45 and upwards who died per thousand clergymen aged 45 and upwards, as estimated by the Census, and take the corresponding figure in the case of the railwaymen, the death rate of clergymen is greater than the death rate of railwaymen. When you analyse the figures, however, you find that for each age group the death rate of railwaymen is greater than the death rate of clergymen. The reason is that this low group figure to which I have referred is a weighted mean, and it is due to the fact that there are very few old railwaymen left.

Here is another example. This example is taken from a paper by Dr. Snow on "Trade Forecasting", published in the *Statistical Journal* of May, 1923. Dr. Snow is a well-known statistician, and he is particularly interested in the question of leather.

| Date. | September, 1921. | October, 1921. |
|---|---|---|
| Dry and Salted Hides - | £2·36 per cwt. | £2·67 per cwt. |
| Wet Hides - - - | £5·29 per cwt. | £5·36 per cwt. |
| All Hides - - - | £4·11 per cwt. | £3·83 per cwt. |

Looking at the figures for the average price in pounds sterling per hundredweight of hides imported into England in the two successive months indicated, Dr. Snow found that the average price went down. He knew from his own experience that the average price had gone up in every kind of leather with which he was acquainted, and he tried to get to the bottom of it, and here is the explanation. Hides are divided into two classes, dry and salted, and wet. For the dry and salted hides, the cheaper hides, the price had gone up, and for the wet hides imported the price had similarly gone up. But again

we have a weighted mean. A much larger proportion of dry and salted hides had come in in the month of October than in September, and in consequence the average price had been pulled down, so that anyone looking at the group statistics would have said: "Oh, hide is selling cheaper", whereas actually every kind of leather was getting dearer but more cheaper leather was being imported. That is a fairly simple illustration which can be shown to a class of average boys soon after they have had the first ideas of averages.

(2) In graphs "Statistics"; in arithmetic "Significant Figures". All measures are subject to error. In statistical returns these are often compiled from data that are subject to very considerable errors and, except for accountants' figures, the total compiled is very seldom correct to more than three significant figures, and often not even that. No statistician would ever make a statement for which *The Spectator* ignorantly derided them (see Gleaning 848, *Math. Gazette*, XVI, No. 218, May 1932, p. 95): "Exactly 1,335,617,903 pounds of chocolates and candies were consumed in the U.S.A. in 1930". Yet often such data are given to our pupils to graph. Furthermore, particularly in historigrams, it is often not clearly stated whether the measure recorded is a time-point of observation, as in the case of a Census of Population record, or a time-accumulative observation, as in the case of total exports to a country. In either case, there are errors of unknown magnitude, and no pupil should be expected to put graphs to such points. Whatever they do, they would make unfounded assumptions. If, to follow the book already referred to on its p. 158, they join the plotted points by straight lines, they assume linear variation between successive dates and a discontinuity in the gradient. Linearity is of course most unlikely, except as a first approximation; and discontinuity we may say a practical impossibility, though it may occur in government statistics as the result of a revolutionary legislative change. As this last would be as likely to affect the continuity of the graph itself and not therefore likely to be the sort of thing we would give our novices we should rule out all "dog-tooth" graphs. If we smooth by arbitrary graduation rules equivalent more or less to a least square method we again really assume errors of a certain relative magnitude with other approximations as well. I think it very desirable, therefore, that for school work we should discontinue all statistical graphs where the points do not give a smooth curve approximating to some empirical law. Thus I should allow time of sunrise and date, or time of sunrise and latitude, expectation of life, Perry's law (record and distance), moon's age and height of tide, but not epoch with census, imports, crime, unemployment, price index numbers and similar data.

(3) In Probability, the statistical steadiness of large numbers may be used to give us problems, and the "Either or" and "As well as" problems, as Arne Fisher calls them, may be dealt with on Sheppard's lines. Probability on statistical lines has been dealt with here both by Dr. Sheppard and Mr. Hope-Jones and I need hardly speak

on this now, though I feel that I might say that for the type of pupil that I have in mind very little will be comprehensible.

(4) With regard to correlation, I feel that the computations are utterly out of the depth and are far too time-consuming for any work to be done on these lines. I think, however, that an elementary study of a correlation table on Galton's lines is desirable as indicating that there is a stage where statistical variables are neither independent nor mathematical functions of one another.

In the main, however, with regard to the question of correlation as of the study of distributions, I feel that the practical work involved in mere counting, plotting and computation is so heavy that we cannot hope to do it in schools. It is for this reason that I have found Nunn's suggestions of little value.

To conclude, I feel that we cannot honestly establish a case for statistics to oust any other recognised mathematical subject from the time-table, that the mathematical syllabus contains little that can be replaced by statistics as an alternative, that the average pupil is too immature to grasp statistical ideas and the average teacher too ill-equipped, and that all that can be done at present is to modify present mathematical teaching in certain directions. I do not suggest that for exceptionally bright pupils such as I meet with only occasionally the progress through the time-table and the early outstanding intellectual capacity may not be sufficiently pronounced that suitably qualified teachers may not find a statistical course, especially for say Sixth-form general cultural work, a possibility. A course in statistics cannot usually be provided, and I think statistics as such can have no place in the ordinary secondary school programme. On the other hand, a statistician's outlook can very profitably colour our treatment of certain elementary topics in school mathematics.

**Mr. C. O. Tuckey** (Charterhouse): I feel that I am here somewhat under false pretences, because my knowledge of the subject is not extensive and the frequency with which I have taught it is very slightly correlated with the high opinion I hold of it, but I am encouraged to speak by the fact that what I say will be very different from what has been said to you. I shall not be repeating what has been said; I shall disagree with it entirely.

I think the most useful thing I can do is to tell you of my very small experience of teaching the subject and then suggest from that the classes of pupils to which I think it is practical politics to try to teach the subject.

I have taught it to two types of pupils, one being the occasional mathematical specialist, who for some reason or other wanted it or seemed to have time to study the subject. With this type of pupil I have used Yule's book, and we have always enjoyed it very much, except the first part on Attributes, and, acting on the excellent principle that anything I found hard was too hard for my pupils, I got on very well. That is one kind of experience I have had.

The other experience is somewhat different. At Charterhouse some time ago we had a scheme by which, in the lower middle school,

we allotted a couple of periods a week to be used in going beyond the ordinary course of certificate work. These periods were to be labelled " Higher Mathematics ", and each master had a free hand to select what he would teach his form as higher mathematics for a couple of periods a week. We all followed our own inclinations. During most of the time that I did this I took statistics, and I took it with a form in no way gifted in mathematics. The statistics we did was somewhat less in amount than what is indicated in the Arithmetic Report. We did frequency curves and the different types of average ; the mean, the median and the mode. I should like to suggest that, with a low form, the fact that there is a mode which is not the same as the mean is an idea which is a considerable novelty to many boys and is a very valuable one. Then we did measures of dispersion. I think also that the idea that the size of the measure of dispersion makes a difference in how far the mean is really representative, is an idea which is useful and valuable for the stupid boy. Then we did a very little correlation ; we went as far as getting two lines of regression. That was about all we could do. That was a course that lasted about two hours a week for one term, with dull boys, and it was at any rate a very amusing and pleasant variation from the ordinary certificate work. That is the sum total of my experience in the subject.

On the strength of that I would say that there are two or possibly three types of pupils in schools to whom it is practical politics to teach this subject. One is the mathematical specialist who does not belong to the class of the scholarship candidates. It is not practical politics to put statistics into the scholarship candidates' course unless it is put there for you. The same applies to the course of the Woolwich candidate and the future engineer ; these have got plenty to do already. But there still remains a certain number of mathematical specialists who have specialised in mathematics because they prefer that subject, but who have not an engineering career or a scholarship examination in front of them. They are perhaps going into business. I am not speaking of the actuary, for whom statistics is a professional subject, but of the future accountant or the future business man. For that type of mathematical specialist I believe that the theory of statistics is one of the most valuable subjects we can possibly teach him, and it is perfectly practical politics to teach him that now. There being no examination in front of the boy, there is nothing to prevent the master teaching him what he pleases and what he thinks is most valuable educationally, and I would put in a plea to all teachers of mathematical specialists to consider the possibility of starting at once to teach the theory of statistics to those of their mathematical specialists who belong to the class I have mentioned.

Then, secondly, there is the much more elementary work which I took when I was teaching the middle school division and which I have just described to you. That seems to me very appropriate for the use that I made of it, that is to say, for a division in which you have a couple of hours per week to spare from the ordinary grind.

Of course, there are many other subjects you can use in the same way, and it is very rare to get the two hours a week. But, if you get that curious combination, it is a very nice subject to take.

Then, further, there is one other possibility, namely, as post-certificate work. If there are pupils who continue their mathematics not as specialists but for a short time per week, say four or five hours a week, after they have done the ordinary certificate course, and who are simply keeping up their mathematics because they wish to do so, there again you have a chance of taking this subject. In that case it is difficult, to my mind, to say that other subjects are more valuable, because presumably such pupils are not very able mathematicians, so that they are not likely to benefit by small scraps of the scholarship course, or at any rate they are less likely to benefit by it. It is better not to take subjects which are just detached from the scholarship course and are obviously incomplete, but to take something which is different, and, if possible, to take something which has perhaps more application to the reading that the pupils may do in other subjects. For instance, historians may be doing mathematics as well as taking economics as part of their main subject, and they will not understand their economics unless they have some knowledge of statistical method.

That is really all I have to say, but I should like to emphasise strongly, in opposition to the former speaker, that I have a high opinion of statistics as a very valuable subject to teach boys and a very pleasant one to teach, and the only difficulty is that it seems not common to find the occasion to fit it in. When you can fit it in, my advice is to do so.

**Dr. J. Wishart :** I doubt if it is altogether wise to ask someone who is a specialist in this subject to contribute to the discussion. There is a real danger that such a one might out of genuine enthusiasm for the study of statistics recommend its wholesale introduction into the schools. But I think on the whole I should be inclined both to agree and to disagree with Mr. Sandon. It is difficult to see exactly how statistics is going to be brought in, and it is open to question whether it would in all cases be advantageous. Mr. Sandon's general summing up was this, that the school outlook would be all the better for a comprehension—I think he meant on the part of the teachers—of statistical ideas, and that the prime object of teachers might well be to indicate that there are these problems and to leave the students to take them up later.

I have had only a very little experience of school teaching and none at all of trying to teach statistics in schools, but perhaps a word or two on what happens in that respect at the universities might interest you. At most universities statistics is not at the moment any recognised part of the career of the mathematical scholar. It is usually taught in conjunction with other subjects to students of economics, actuarial science, and so on. Cambridge has, however, recently started an advanced course for mathematicians, and it is quite interesting to me to see the number of people who, without any particular urge, have come up and said that they would like to take

the course. That is one side, concerning the mathematical specialist, and I have no doubt that many of them will in the future become teachers of mathematics.

On the other side, however, there is a large and ill-defined class of students in the biological sciences. They are usually not mathematicians at all; we find them saying that mathematics was their worst subject at school. They find, however, that in experimental science they need more and more nowadays to know something about statistical ideas, and, as a consequence, they feel themselves rather badly equipped. They wish they had learned more about statistics before they came up, and, when they do take up the study of statistics, they like it.

In fact, you see, one has to teach statistics in two ways: one with as much mathematics as you like, and the other where the less said about mathematics the better. It is possible, I think, though sometimes rather difficult, to get round the mathematics to a considerable extent, and it is encouraging to see how these people find a real interest in their statistics, especially when it is tacked on to the particular job they are doing.

School education should, of course, above all things, keep abreast of the times, and I do not know how far it is realised in schools that the research institution has made enormous strides, either in or apart from the university, and that scientific research as a career is a very big thing indeed. There are many scientists who are exceedingly badly equipped as far as statistics are concerned, and it is wonderful how many mistakes in interpretation are made by research workers. A theory which seems rather wonderful on paper might often be scotched at birth if the experimental results were subjected to the strict scrutiny of statistical analysis.

I think there is something to be said along the lines of what Mr. Sandon indicated, and that, rather than teaching statistics as a separate subject or as an alternative to some part of the pupil's mathematical course, we should teach, as part of the mathematical course, some of the ideas in connection with statistics which will later on be useful. That means that a good deal can be done in the way of expanding one's ordinary arithmetical ideas about averages, and, while I appreciate what Mr. Sandon said about giving a warning that the weighted mean may sometimes tell the wrong kind of story, I wish he had given another example to show how the unweighted mean, which is so often used, may be equally fallacious. The weighted mean is not to be altogether condemned by the examples Mr. Sandon put on the blackboard; I should say that in most cases it is the right mean to take. Children can be taught to do really accurate work—in fact are, I hope, taught that now. This is a machine age, and the use of the calculating machine is spreading, but the fact that calculating machines are used in a great deal of elaborate work does not mean that accuracy goes by the board, or that it is no longer necessary for students to learn to do accurate work. It would not be a bad plan, perhaps, if a school could get hold of a calculating machine and teach the students to use it,

because such machines are used so much nowadays in business houses.

With regard to graphical work, I did not quite follow what Mr. Sandon said about the kind of graph he put on the blackboard. My experience of schools is that the only kind of graphs we drew were smooth parabolic curves, and so on, or graphs in connection with Boyle's Law in our physics course, where we plotted $P$ against $1/V$ and were told that the result ought to be a straight line, though we did not always find that it was a straight line. I think the pupil gets the idea that a graph is either a straight line or a smooth curve and that at school he has no sort of idea of the scatter of points that is so often the experience of any worker in the biological field. I think he should have the idea that there is this cluster of points, but that there is nevertheless some definite backbone, as it were, to that cluster of points which probably represents the true law. It may be going too far to ask them to fit curves to the data by least squares, although the more advanced student may profit by this.

I am not sure that I have said anything very definite. It is not only from the vocational point of view that statistics are valuable; they are valuable also from the cultural point of view. Really all I have given is just one or two hints on certain parts of the arithmetic and algebra courses which might be a help. If there was any suggestion of adding to the already heavy burden of the mathematical student, I should be inclined to say, "Hold off a bit longer", but it is probable that there are certain ways in which the reorientation of one's teaching could take place.

I would just conclude by saying that there seems to be to-day a better recognition, in the daily Press and elsewhere, of what statistics means, that it is something more than the tabulation of figures, and the time is ripe for trying to train our pupils and our students to have something like a statistical outlook and to be able to make rational deductions from the figures they examine. They have plenty of figures nowadays in the newspapers—unemployment returns, census figures, and so on—on which they can practise. Those who saw the original note in the *Spectator*, which was referred to in the *Gazette*, would see that there was one statistician who had the temerity to reply to the article, which consisted rather of a sneer at the statistician. I do not know whether it was meant in fun or not, but there is a good deal of that kind of thing. It is commonly said that one can prove anything by statistics. Well, one can go through a certain line of argument, but the real function of statistical knowledge is to enable one to give rational interpretations to series of data.

I have read the little description which is given in *The Teaching of Arithmetic* as to what statistics ought to be taught in a school course. There are people who occasionally do find an odd pupil who does not have to be crammed for some examination but can be taught for the benefit of his mind, and it is very interesting that there are teachers who are making this experiment. One can only hope that more and more teachers, either through training colleges or through universities, will have the opportunity of learning more

statistics and so will be better equipped to introduce the subject. I would suggest that one addition might be made to the books suggested, and that, besides the use of books like the excellent text-books of Yule and Caradog Jones, something also should be done in the way of studying a research workers' manual, such as that of Dr. R. A. Fisher (*Statistical Methods for Research Workers*—Oliver and Boyd), because the new books deal with developed points of theory which are hardly touched on in the older books. If we are beginning the study of statistics in schools, I think we ought to present points of view and theories as they are understood to-day and not as they were understood, say, twenty years ago. Of course, it is a theory which has developed very rapidly, and I hope our knowledge may be consolidated a little in the near future, but there is a great deal more known to-day about the ordinary theories that are applied to the experimental data of the biologist, and so on, than was the case a good many years ago.

**Mr. R. M. Wright** (Winchester): I should like first of all to say that I disagree almost entirely with what Mr. Sandon said at the beginning and agree almost entirely with what Mr. Tuckey said to follow, and that should by itself provide a fruitful basis for discussion when I have finished.

With regard to the experience I have had in dealing with the subject of statistics, I first met the subject at Cambridge, when, after having finished my mathematics, I was, comparatively speaking, amusing myself with other subjects. I heard some lectures by Mr. Udny Yule on the subject of statistics and at first it appeared to be rather elementary arithmetic, but then I discovered that it had a great deal of mathematical fascination about it as well as having obvious practical applications. In my teaching experience I have had the opportunity, in the course of the last eight years or so at Winchester, of teaching the subject occasionally to boys who certainly are of the average or better intelligence, some of whom had not yet reached their school certificate examination and some of whom had passed it, but all of whom, anyhow for the time being, did not have to bother at all about any external examination. That, of course, is one of the great advantages of the subject—that, in teaching it in a school, either to boys or to girls, you can be fairly certain that the unfortunate children are not going to be examined about it, so that it does not matter whether you teach it well or badly. The kind of knowledge with which these boys were already equipped varied a great deal. Some of them knew little more than how to draw a graph and that sort of thing, some had done a certain amount of the calculus, and some knew a little bit about $e$; if they did, so much the better. But, even if they knew nothing about the calculus or $e$, there was always in the subject a certain amount one could do which would appeal to boys of average intelligence and average knowledge, both from its interest and from the fact that the boys were doing something themselves.

I may say first that I have always followed the course that is outlined in Nunn's most admirable book, *Exercises in Algebra*, a copy of

which I have just borrowed to see if there is a new edition. As far as I can see, there is not. The statistical section seems to me to be rather a dangerous section to follow blindly, but it does give an outline of an extraordinarily interesting course. It starts with a section dealing with the drawing of frequency curves. In that connection the great thing, I think, in taking the subject with boys or girls, is to collect statistics from them, and you can then build up quite instructive graphs. I started at the beginning of the school term, when the boys had a fair amount of money, and told them to take the money out of their pockets and put it on their desks and arrange it in accordance with the dates on the coins. They then got little piles of coins in front of them, perhaps starting as far back as 1872 and possibly—though, on the whole, it was not likely—going on as far as 1932. We then had the beginning of a frequency distribution, with sixty different classes of years, 1872 to 1932, but we got an unwieldy distribution in that way. Sixty classes are too many to handle in one frequency table. It is clearly better to group them, and you at once get two or three examples out of that same collection of coins. First of all you can try to group them by years and make up the frequency columns with the year and the number of coins belonging to that year in the room, and you can collect the figures for your frequency table quite quickly in that way. You can then start grouping the coins in periods of five years, starting from 1932-1928, and you will at once get your column graph looking something like this :

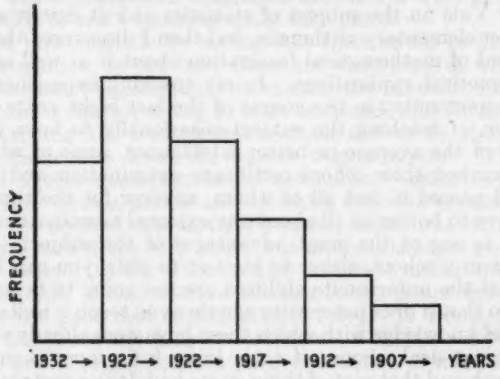


FIG. 1.

Of course, you have to make it clear whether you mean 1932 down to 1928 inclusive or down to 1927 inclusive, as the case may be.

If you then make a different grouping, say 1932-1923, 1922-1913, 1912-1903, you get a column graph probably looking like Fig. 2.

You would get a distribution of the type that is fairly common, for instance, in connection with the distribution of incomes among the British population. If you turn to the statistical abstract for

the United Kingdom and look up the distribution of incomes amongst those paying income tax, you find the frequency curve will be of that type.

That is one obvious example that you can get out of the pockets of the boys, and you can get a similar one, and one of great interest, from the frequency distribution of the ages of your division. That brings in at once the problem of how you are going to define your class and class-interval precisely, because you are no longer dealing with whole numbers but with numbers like 13 years 7 months,

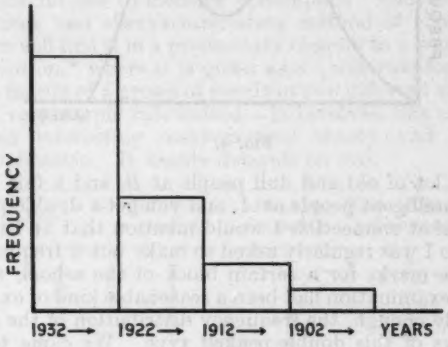


FIG. 2.

15 years 6 months, and so on. You have to define your class carefully. If you define it as over 15 and under 16 you have to decide what is going to happen to a boy who is 15 that day ; points like that crop up, showing how, in making up the frequency table, you have to be very careful in defining the class-interval precisely and deciding to which class any one particular observation may belong. That frequency distribution may or may not give you the normal distribution. In a good many classes it would give you a distribution like this :

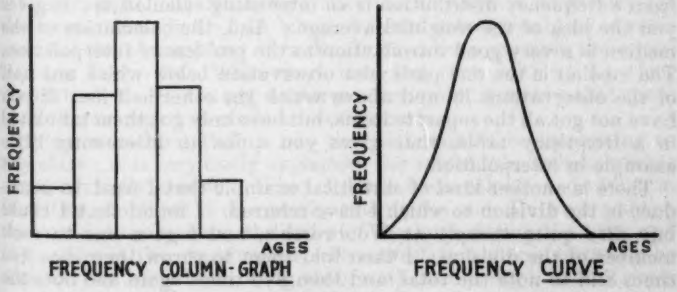


FIG. 3.

Actually, however, in the type of division with which I was dealing, the grouping was nearly always like this :

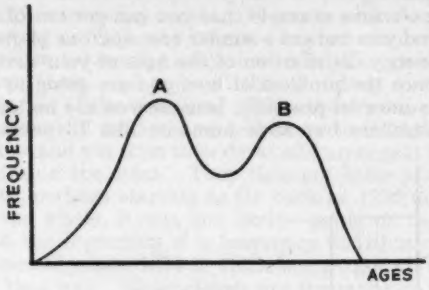


FIG. 4.

You get a lot of old and dull people at $B$, and a fair number of young and intelligent people at $A$, and you get a double-peak distribution. In that connection I would mention that at Eton quite a long time ago I was regularly asked to make out a frequency distribution of the marks for a certain block of the school, to find out whether the examination had been a reasonable kind of examination, and, curiously enough, the frequency distribution of the marks was nearly always of this double-peaked type. We came to the conclusion, however, that it was not due to the faultiness of the examination, but to some kind of element occurring in the group examined; the group used to contain a certain number of young and clever people and a rather larger number of older and less clever people. That was the kind of distribution that always used to come out in the frequency distribution of the marks in the middle division at Eton, dealing with about two hundred observations.

Drawing the column graphs and the frequency curves arising from them is, comparatively speaking, child's play, but it is quite interesting work ; you can go on then to deal with the definition of the mean, the median and the mode. The actual calculation of the mean from a frequency distribution is an interesting calculation ; it gives you the idea of the weighted average. Also, the calculation of the median is a very good introduction to the problem of interpolation. The median is the one particular observation below which one half of the observations lie and above which the other half lie. If you have not got all the separate items, but have only got them tabulated in a frequency table, that gives you quite an interesting little example in interpolation.

There is another kind of statistical example that I used to introduce in the division to which I have referred. I found that I could buy dice quite cheaply at Woolworth's, and I gave one to each member of the division. I then told them to throw their dice five times and to note the total, and then five times again and note the total, and so on, until we had collected about two hundred obser-

vations of the total throw of five dice. Then, of course, you get an almost perfect example of the normal probability curve. There is no doubt that in practice, if you carry out an experiment of that sort, you get a very beautiful curve, coming out pleasingly close to the shape that theory will predict that it should have.

I have very little time left, but I should like to refer at least to two other rather more difficult items in the course. Mr. Sandon said that correlation was too difficult a subject to be taught in schools. The whole theory of correlation is, of course, a difficult subject, but I certainly should like to mention Spearman's "Foot-Rule", which is a very quick and always interesting method of comparing two orders. You will find it in a preliminary chapter in a book on experimental education,* where it is given as a quick rule for comparing the relative merits of a group of pupils in two different subjects, and it is a very very simple rule indeed. It involves, like all statistical methods, an interesting mathematical theory and quite good practical application. It simply depends on this.

| Pupil. | Rank in Algebra. | Rank in Geometry. | Loss of Rank in Geometry. |
|---|---|---|---|
| *A* | 1 | 3 | 2 |
| *B* | 2 | 1 | — |
| *C* | 3 | 4 | 1 |
| *D* | 4 | 2 | — |
| *E* | 5 | 5 | — |

If you have five pupils and you write down their order in two subjects, the total loss of rank is found by counting up how many places have been *lost* by each member of the group. This is very elementary to all of you, perhaps, but I should like to make it quite clear, because I think it is rather useful. Here you have a total loss of rank of 3. The *average* loss of rank, if you take the total number of permutations of $n$ pupils and take the average loss of rank that will occur if all the possible permutations occur, is $(n^2-1)/6$. It is a pleasant and not very difficult piece of algebra to prove this.

Spearman's Foot Rule says that the correlation coefficient for these two orders is $1-L/A$, where $L$ is the actual total loss of rank and $A$ is the average loss of rank. In the above example the correlation is $1-\frac{3}{4}=\cdot25$. When there is no loss of rank there is perfect correlation, 1. If you get negative correlation, it means that more than the average loss of rank has occurred.

Spearman's Foot Rule is one that is bound to be interesting to any class; it is very easily explained, the actual rule of thumb being extraordinarily simple. When I have been taking a division in statistics and there is an examination at the end, I always publish the correlation between the order of the boys during the term and their order in the examination. The correlation should normally be high if they have been doing their work in the right kind of way.

* *Experimental Education*, by Robert R. Rusk (Longmans, Green & Co., 1919).

I always regard a high correlation as a good sign for the division and for myself, and a low correlation as a bad sign.

That is one of the slightly more advanced pieces of work. Another which I think I must mention is only useful if you have pupils capable of dealing with $e$. If they know merely the value of $e$ they can be given this as a rule of thumb: the frequency-curve of a normal distribution can be expressed by the formula

$$y=\frac{N}{\sigma\sqrt{(2\pi)}}e^{-x^2/2\sigma},$$

where $N$ is the total number of observations made and $\sigma$ is their "standard deviation". The proof that this is the theoretically correct formula is likely to be too difficult for most pupils.

I have said nothing about the calculation of standard deviation, which again involves a very pleasant little bit of algebra; it is a very good test of accuracy to calculate out a standard deviation from a set of observations in a frequency table. I got the idea of this example from an experiment described in Nunn's book, about an engineer, Mr. Merriman, who erected a target with 11 horizontal lines on it and then fired 1000 shots at the centre line. He took the number of shots falling on each line and found that they obeyed the normal distribution, so that he got a curve very much of this shape:

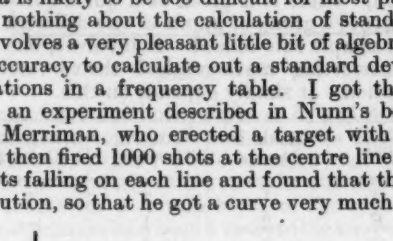


FIG. 5.

The frequency of a deviation $y$ from the centre-band, numbered 6, should theoretically be given by the formula mentioned

$$y=\frac{N}{\sigma\sqrt{(2\pi)}}e^{-x^2/2\sigma^2}.$$

You cannot carry out that experiment in a class room, but you can do an experiment which I find is almost exactly the same and which delights any division on a hot summer afternoon, because it is one they do for their own amusement. You take a piece of paper and rule lines on it and then take your compass and shoot at the bull. Every boy has a chance of doing that fifty times, and you then collect the results. I have found that you do get sometimes a very good normal distribution; I never quite know why. Theory

seems to predict it supposing you have the same man carrying out the experiment in exactly the same way each time. If you have twenty different people carrying out the experiment in different ways with different degrees of skill in aiming, you would not expect the distribution to be particularly normal. The last time I carried it out it went very badly wrong, but I found that was entirely due to one boy, who was not aiming at the bull in a way in which he had a good chance of hitting it. He had therefore got a very much worse kind of result on his target than the results that the rest of the division had got on theirs, and the interference of that one particular score spoiled the normality of the distribution. Otherwise the actual results agreed very closely with those predicted by theory.

I am afraid that all I have said is very sketchy, but I think I have achieved my object if I have succeeded in pointing out there are methods here that are bound to be amusing, provided one can get rid of any fear of examinations and provided one is reasonably interested in the subject. You can then be quite certain, I think, that any pupils who try to follow the kind of course that I have indicated will be interested, even if at times they get completely out of their depth.

**Professor Hamley** (Institute of Education): I entirely agree with those who desire to bring an element of statistics into the ordinary school course. In working out some experiments on the teaching of mathematics a few years ago, I found that it was possible to introduce some of the concepts of statistics from the earliest years of the secondary school course—from the age of about 11 plus. I was led to this through an effort to unify the whole course of school mathematics. I came to the conclusion that the subject matter of the school course in mathematics should be grouped round four great concepts, namely, "the class", "order", "the variable", and "correspondence" or "the function". I therefore devised a course which would bring in these four concepts as the central themes. We started our work with a discussion of "classes", and our first lesson in algebra was what could be called a lesson in classification. For example: we were given two examination subjects, say arithmetic and English, the results of which were divided into grades $A$, $B$ and $C$. We were told that some boys obtained $A$ grade in both subjects, some $B$ grade in both subjects, and so on. Sufficient data were given for a "manifold classification", to use Yule's term. In this exercise the minimum amount of information was given from which all the cells of the distribution could be filled in. Then questions like the following were asked: "How many boys who did not obtain $C$ in arithmetic did not obtain $B$ or $A$ in English?" This is an excellent problem in elementary logic, and I can tell you that the boys were intrigued by work of this kind. Later we took the boys into the school playground and arranged them "tallest on the right, shortest on the left". The boys then measured their heights and drew a graph to represent the distribution. In this way we were able to get our first picture of a statistical distribution, which we called an "array", because the boys were "arrayed" like soldiers. In subsequent exer-

cises we called this a " variation array ", and made ourselves familiar with such terms as : " median ", " upper limit ", " lower limit ". One other detail is worth mentioning. In our first exercise we numbered the array from the left-hand side. We followed this by numbering the array from the middle. Suppose, for example, that we had thirty boys in the class. In the new numbering No. 15 would be No. 1 ; No. 16 would be No. 2, and so on. Then the question arose : How were we to number the boys on the left of No. 15 ? This led us, of course, to the subject of directed numbers ; the array giving interesting examples of the properties of positive and negative numbers. This was followed by a little more graphical work, until we came to the question of averages, which was introduced in a manner similar to that already indicated to-day. The boys were asked to guess their average height. Knowing the median they were generally able to do this to within a fraction of an inch. They then set to work to find out whether the guess was right or wrong. This they did, not by adding the heights of all the boys in the class and dividing by the number of boys, but by getting the deviations above and below the guessed mean. Then we went on the further graphical work until we came to the frequency histograph. We approached this through ordinary bar or column graphs of discrete statistical data. For our first practice exercises we drew graphs of the exports and imports of a country, and then used the results of a class examination to give us a frequency distribution. Sometimes the results gave us the double-hump type of graph that has been referred to to-day ; sometimes a normal frequency curve. The two types were discussed by the class.

In this way it was possible to introduce, in the normal course of school work, practically all the main concepts of statistics. When we came to correlation we first called the relationship " correspondence " and referred to the " correspondence " between variables. Statistical ideas were also brought out in the physics lessons. For example, when dealing with statistical data from experiments we called the graph of the plotted points " the line of best fit ". Later we were able to define the line of best fit more precisely, and at the end of the course in mathematics to discuss the Gaussian " Method of Least Squares ". As a result of the experience I have had I can assure you that it is possible, in some such way, to bring out practically all the main concepts of statistics naturally and easily in the school course.

I believe that we need three different types of statistical work in schools. We need, as our first type, a general elementary course running through the whole of school mathematics. Secondly, we need a course of statistics for the Classical Sixth boys, who want some additional work in mathematics. In reply to questions from headmasters, this Institute has made this suggestion. We have found that boys become keenly interested in lessons on the main conceptions of statistics during their last year at school. Thirdly, I think we need at the end of the school course a special mathematical treatment of statistics for mathematically-minded boys.

I want to say one word with regard to the reference made to Professor Thorndike's statement that an intelligence quotient of 110 is necessary for the learning of algebra. When I was in New York a year or two ago, a friend of mine had just completed an experiment on teaching the Calculus to a class containing a large proportion of negro girls. The intelligence quotient was in some cases below 95. By following functional methods based on the concepts that I have just mentioned he found that he was able to bring most of his class to a pass standard. Some of the results compared favourably with those of students of higher intelligence who had been brought up on conventional methods. My friend drew the conclusion that an intelligence quotient of 110 is not necessary for the learning of the Calculus.

**Dr. Sheppard**: I would suggest that what has been said on this subject is decidedly encouraging, in two respects. It is encouraging that the work done in the schools is really interesting; both Professor Hamley and Mr. Tuckey have shown that the students take a very strong interest in the particular problems put before them. The other aspect is the study of the subject as a branch of mathematics. I believe that there is still a great deal to be done on the theoretical side in getting improved methods for dealing with large masses of figures and getting results out quickly. I, for one, hope that the study of statistics in schools will continue and increase.

**Mr. Sandon** writes as follows:

I feel gratified that the paper as submitted has led to such an interesting and stimulating discussion, and although the speakers seemed to unite in saying how they disagreed with me, I, for my part, feel that they have simply expressed their approval of what I said. May I sum up?

I was speaking of statistics in a school course, and I carefully pointed out the type of pupil I had in mind. I did not, therefore, attempt to deal with the case of the mathematics specialist, with pre-certificate candidates who have extra periods outside the standard course, or post-certificate candidates, classical Sixth men or intending biologists, for whom, again, some special provision can be made. I had in mind the great mass of the more average pupils.

For any such as these, any work involving any reference to $e$, to least squares or to calculus is out of the question. Similarly, any reference to R. A. Fisher's work seems to me would be incomprehensible. The only way I submit we can use Fisher's work is to use it as a warning of the grave dangers of taking small samples, and in this connection I should like to join issue with Professor Hamley. A class examination for a class of ordinary size in a secondary school has a total population far too small to be discussed by elementary methods. We can get double-hump distributions and all sorts of unusual occurrences. They are, I submit, probably meaningless, and I feel that any statistics we should teach in a mathematics course should be sufficiently rigid as not to have to be unlearned later. Spearman's Foot Rule is a very rough approximation, but it is based, in its application, on unjustified assumptions of the nature of the

distribution. Merriman's results are (I speak without referring to the literature, to which I have not convenient access here) those that were specially discussed by Pearson, where he showed that the goodness of fit of Merriman's figures to a normal curve was infinitesimal.

With regard to Dr. Wishart's comment on my dog-toothed graphs, the proper place, I think, for discussing the scatter of points for a smooth curve is in connection with a discussion of the subject matter which those points exemplify. I think it desirable that the physics student in his physics lesson should have it pointed out that his points do not lie exactly on a smooth curve. The reasons can be better pointed out here than by the mathematician, who is limited to a more general discussion of questions of approximations, as I endeavoured to point out.

I was interested to hear Professor Hamley on the negro girls of I.Q. often below 95 who reached a Pass standard in the calculus. I expect many other members of the Association, in view of what has been said before to them at different times, were as astonished as I.

To conclude, I still claim that Statistics as a rigid mathematical discipline cannot be fitted into a secondary school course such as that through which the great majority of secondary school pupils in the country at present work.

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**922.** You remember the well-known story about Leibnitz. He had booked his passage on a little vessel in order to cross the Adriatic. During the voyage, a storm sprang up and the craft was in grave peril. The other passengers, being creatures of a superstitious turn of mind, began to observe Leibnitz with considerable attention; for, as you will readily imagine, the originator of the Differential Calculus did not look very much like the general run of people, which was a piece of grave imprudence on his part, especially at a time of crisis. This was soon made unpleasantly manifest to him. The pilot, jabbering Italian, had already begun to inform the others that the silent man over there, all clad in black, was undoubtedly a heretic, that it was at his instigation that the elements had put themselves into such a fury, and that it was their obvious duty to throw him overboard if they wished to avert the evil spell and escape with their lives. Leibnitz, being an accomplished linguist, heard and understood, and saw that he must behave like the rest if he was to save his skin. With characteristic presence of mind he pretended to be quite unconscious of the peril that threatened him, and producing a rosary, which by a lucky chance he happened to have in the pocket of his *surtout*, began to tell his beads with every appearance of religious fervour, muttering algebraic formulae instead of prayers. This piece of acting saved him his life. When folly reigns, it is well to act the fool if you want to avoid remark.—Nicolas Ségur, *The Opinions of Anatole France*, p. 9. [Per Mr. V. Naylor.]

**923.** He [a young man, walking after midnight] turned to the right, then spent the next minute wondering which of the faint points of light was the North Star, and the next ten minutes wondering about stars in general. It was, as usual, a cheerless meditation. If you are going to bother about these things, he decided, you have to turn astronomer, to weigh them and measure them, out of sheer self-defence.—J. B. Priestley, *The Good Companions*, ch. 6. [Per Mr. P. J. Harris.]

**924.** . . . the pyramidal summit of Kamet stood out like a geometrical proposition set by some heavenly Pythagoras.—F. S. Smythe, *Kamet Conquered*, p. 285. [Per Mr. H. Berry.]

# THE USE OF SYMBOLS OF OPERATION IN THE SUMMATION OF ALGEBRAIC SERIES.

By Harry Freeman.

1. The ease with which certain series involving finite difference operators can be summed by the method of separation of symbols has long been recognised. It does not appear to be universally known, however, that many forms of algebraic series which at first sight do not seem to lend themselves to this treatment can be summed very simply by the application of operators.

2. A few simple definitions suffice for the basis of the method. These are :

(*a*) $Eu_x = u_{x+h}$ ; $E^n u_x = u_{x+nh}$.

For example, $Ex^2 = (x+h)^2$, $E^n a^x = a^{x+nh}$.

Or, if the interval of differencing be unity,

$$Ex^2 = (x+1)^2, \quad E^n a^x = a^{x+n}.$$

(*b*) $$\Delta u_x = Eu_x - u_x = (E-1)u_x,$$

where $\Delta$, $E$, 1, are three symbols of operation.

$$\Delta^n u_x = (E-1)^n u_x,$$

and $(E-1)^n$ can be expanded by the binomial theorem.

(*c*) $\Delta$ and $E$ obey the simple algebraic laws of distribution, commutation and indices. Thus

$$\Delta(u_x + v_x + w_x + \dots) = \Delta u_x + \Delta v_x + \Delta w_x + \dots ;$$

$$\Delta c u_x = c \Delta u_x ;$$

$$E^n(E^m)u_x = E^{m+n}u_x ;$$

$$\Delta(\Delta^{-1})u_x = \Delta^0 u_x = u_x.$$

3. We may separate the symbols from the functions on which they operate, as in 2 (*b*) above. The following example is typical.

$$\begin{aligned} u_0 + u_1 + u_2 + \dots + u_n &= u_0 + Eu_0 + E^2u_0 + \dots + E^n u_0 \\ &= (1 + E + E^2 + \dots + E^n)u_0 \\ &= \frac{E^{n+1} - 1}{E - 1} u_0 \\ &= \frac{(1+\Delta)^{n+1} - 1}{\Delta} u_0 \\ &= \frac{1}{\Delta}\left\{1 + (n+1)\Delta + \frac{(n+1)n}{2!}\Delta^2 + \dots + \Delta^{n+1} - 1\right\} u_0 \\ &= (n+1)u_0 + \frac{(n+1)n}{2!}\Delta u_0 + \dots + \Delta^n u_0. \end{aligned}$$

4. The application of the method to algebraic series is best illustrated by an example.

*Example* 1. If $(1+x)^n = c_0 + c_1x + \ldots + c_nx^n$, $n$ being a positive integer, find the value of

$$(n-1)^2c_1 + (n-3)^2c_3 + (n-5)^2c_5 + \ldots .$$

(Hall and Knight, *Higher Algebra*, p. 339, Ex. 21.)

$$\begin{aligned}
&(x-1)^2c_1 + (x-3)^2c_3 + (x-5)^2c_5 + \ldots \\
&= (E^{-1}c_1 + E^{-3}c_3 + E^{-5}c_5 + \ldots)x^2 \\
&= \tfrac{1}{2}\{(1+E^{-1})^n - (1-E^{-1})^n\}x^2 \\
&= \tfrac{1}{2}\{(\Delta+2)^n - \Delta^n\}E^{-n}x^2 \\
&= \tfrac{1}{2}\{2^n + n \,.\, 2^{n-1}\Delta + \tfrac{1}{2}n(n-1)\,2^{n-2}\Delta^2\}(x-n)^2
\end{aligned}$$

(since third and higher orders of differences of $(x-n)^2$ will be zero)

$$\begin{aligned}
&= \tfrac{1}{2}\{2^n(x-n)^2 + n \,.\, 2^{n-1}\Delta(x-n)^2 + \tfrac{1}{2}n(n-1)\,2^{n-2}\Delta^2(x-n)^2\} \\
&= \tfrac{1}{2}\{2^n(x-n)^2 + n \,.\, 2^{n-1}(2x-2n+1) + \tfrac{1}{2}n(n-1)\,2^{n-2} \,.\, 2\}
\end{aligned}$$

which, when $x=n$, becomes

$$\begin{aligned}
&\tfrac{1}{2}\{n \,.\, 2^{n-1} + n(n-1)2^{n-2}\} \\
&= 2^{n-3}n(n+1).
\end{aligned}$$

Note that this proof can be shortened by the use of " differences of zero " :

$$\Delta 0^2 = 1 ; \quad \Delta^2 0^2 = 2 ; \quad \Delta^n 0^2 = 0 \text{ if } n > 2 ;$$

and we have at once

$$\tfrac{1}{2}\{2^n \,.\, 0^2 + n \,.\, 2^{n-1}\Delta 0^2 + \tfrac{1}{2}n(n-1)\,2^{n-2}\Delta^2 0^2\},$$

which reduces to $2^{n-3}n(n+1)$ as before.

Example 18 on the same page of Hall and Knight's book can be done in a similar way :

*Example* 2. Prove that

$$n^3 + {}^nC_2(n-2)^3 + {}^nC_4(n-4)^3 + \ldots = n^2(n+3)2^{n-4}.$$

The series reduces to $\tfrac{1}{2}\{(\Delta+2)^n + \Delta^n\}0^3$, which on evaluation produces the required result very simply.

*Example* 3. Find the value of

$$n^n - n(n-2)^n + \frac{n(n-1)}{2!}(n-4)^n - \ldots .$$

We have $2^n \,.\, x^n - n \,.\, 2^n(x-1)^n + \dfrac{n(n-1)}{2!}2^n(x-2)^2 - \ldots$

$$\begin{aligned}
&= 2^n(1-E^{-1})^n x^n \\
&= 2^n \,.\, \Delta^n \,.\, E^{-n}x^n \\
&= 2^n \,.\, \Delta^n(x-n)^n \\
&= 2^n \,.\, n! \quad \text{for all values of } x.
\end{aligned}$$

Putting $x = \tfrac{1}{2}n$, we have $2^n n!$ as the value of the given expression.

5. If $x(x-1)(x-2)\dots(x-n+1)$ be denoted by $x^{(n)}$,

$$\Delta x^{(n)} = (x+1)x(x-1)\dots(x-n) - x(x-1)\dots(x-n+1)$$
$$= nx^{(n-1)}.$$

Similarly,
$$\Delta^2 x^{(n)} = n(n-1)x^{(n-2)}$$
$$\dots\dots\dots\dots\dots\dots$$
$$\Delta^n x^{(n)} = n!$$

Again, if $1/\{x(x+1)\dots(x+n-1)\} \equiv x^{(-n)}$, then, as above,

$$\Delta x^{(-n)} = -nx^{(-n-1)},$$
$$\Delta^2 x^{(-n)} = n(n+1)x^{(-n-2)}, \text{ and so on.}$$

The use of this "factorial" notation enables another type of series to be summed with little difficulty.

*Example* 4. Sum the series

$$\frac{1!}{a+1} + \frac{2!}{(a+1)(a+2)} + \dots + \frac{n!}{(a+1)(a+2)\dots(a+n)}.$$

(Hall and Knight, *Higher Algebra*, Misc. Ex., No. 274 (2).)

If $S$ denote the given series,

$$\frac{S}{a} = \frac{1!}{a(a+1)} + \frac{2!}{a(a+1)(a+2)} + \dots + \frac{n!}{a(a+1)(a+2)\dots(a+n)}$$
$$= 1!\,a^{(-2)} + 2!\,a^{(-3)} + \dots + n!\,a^{(-n-1)}$$
$$= -\Delta a^{(-1)} + \Delta^2 a^{(-1)} + \dots + (-)^n \Delta^n a^{(-1)}$$
$$= \{-\Delta + \Delta^2 - \Delta^3 + \dots + (-)^n \Delta^n\} a^{(-1)}$$
$$= -\Delta \frac{1-(-\Delta)^n}{1+\Delta} a^{(-1)}$$
$$= -\Delta E^{-1}\{1-(-\Delta)^n\} a^{(-1)}$$
$$= -\Delta \{1-(-\Delta)^n\} (a-1)^{(-1)}$$
$$= \{1-(-\Delta)^n\} (a-1)^{(-2)}$$
$$= \frac{1}{(a-1)a} - (-)^n \Delta^n (a-1)^{(-2)}$$
$$= \frac{1}{(a-1)a} - (-)^{2n} (n+1)!\,(a-1)^{(-n-2)}$$
$$= \frac{1}{a-1}\left\{\frac{1}{a} - \frac{(n+1)!}{a(a+1)\dots(a+n)}\right\}.$$

Thus

$$S = \frac{1}{a-1}\left\{1 - \frac{(n+1)!}{(a+1)(a+2)\dots(a+n)}\right\}.$$

*Example* 5. Evaluate

$$\frac{1}{x}-\frac{n}{x(x+1)}+\frac{n(n-1)}{x(x+1)(x+2)}-\ldots+\frac{(-)^n n!}{x(x+1)\ldots(x+n)}.$$

This series is $x^{(-1)}-nx^{(-2)}+n(n-1)x^{(-3)}\ldots+(-)^n n!x^{(-n-1)}$

$$\begin{aligned}&=x^{(-1)}+\frac{n}{1!}\Delta x^{(-1)}+\frac{n(n-1)}{2!}\Delta^2 x^{(-1)}+\ldots+\frac{n!}{n!}\Delta^n x^{(-1)}\\&=(1+\Delta)^n x^{(-1)}\\&=E^n x^{(-1)}\\&=(x+n)^{(-1)}\\&=\frac{1}{(x+n)}.\end{aligned}$$

6. If $E_1$, $\Delta_1$ operate on $u_x$ alone, and $E_2$, $\Delta_2$ on $v_x$ alone, we may write

$$\begin{aligned}\Delta u_x v_x&=u_{x+1}v_{x+1}-u_x v_x\\&=E_1 u_x \,.\, E_2 v_x-u_x v_x\\&=(E_1E_2-1)\,u_x v_x.\end{aligned}$$

$\Delta^n$ can therefore be expressed as a series in $E_1$ and $E_2$ or, by the use of the relations $E_1\equiv 1+\Delta_1$, $E_2\equiv 1+\Delta_2$, in terms of $\Delta_1$ and $\Delta_2$. Examples of the type given below can be solved very simply by the use of these operators.

*Example* 6. Find the value of

$$xy-(r+1)(x-1)(y-1)+\frac{(r+1)r}{2!}(x-2)(y-2)-\ldots.$$

The series may be written

$$\begin{aligned}&xy-(r+1)\,E_1^{-1}x\,.\,E_2^{-1}y+\frac{(r+1)r}{2!}E_1^{-2}x\,.\,E_2^{-2}y-\ldots\\&=(1-E_1^{-1}E_2^{-1})^{r+1}xy\\&=(E_1E_2-1)^{r+1}E_1^{-(r+1)}E_2^{-(r+1)}xy\\&=\Delta^{r+1}(x-\overline{r+1})(y-\overline{r+1})\quad\text{since }\Delta\equiv E_1E_2-1\\&=0.\end{aligned}$$

7. The methods outlined above are applicable when the interval of differencing is other than unity. Should the series involve functions of the form $a+nb$ instead of the simple $a+n$, they can be dealt with on exactly similar lines, provided that it is borne in mind that the interval of differencing is $b$ instead of unity. A simple example will make this clear.

*Example* 7. Find the value of

$$a^{n+1} - n(a+b)^{n+1} + \frac{n(n-1)}{2!}(a+2b)^{n+1} - \frac{n(n-1)(n-2)}{3!}(a+3b)^{n+1} - \dots .$$

Taking the interval of differencing to be $b$,

$$\begin{aligned} S &= (1 - {}^nC_1E + {}^nC_2E^2 \dots)\, a^{n+1} \\ &= (1-E)^n a^{n+1} \\ &= (-)^n \Delta^n a^{n+1}. \end{aligned}$$

Now $a^{n+1} = a^{(n+1)} + \frac{1}{2}n(n+1)a^{(n)} + \text{terms in } a^{(r)} \quad (r<n)$.

So $\Delta^n a^{n+1} = (n+1)!\, b^n a^{(1)} + \frac{1}{2}n(n+1)\, b^n n!$

$$= \tfrac{1}{2} b^n (n+1)!\,(2a+nb),$$

and $$S = \tfrac{1}{2}(-b)^n (n+1)!\,(2a+nb).$$

8. One of the chief advantages of the method lies in its application to the solution of questions involving the coefficients in the expansion of $(1+x)^n$. While the algebraic solutions are often long and cumbersome the finite difference methods are admirably brief. For example, the value of the well-known expression

$$\begin{aligned} &\frac{c_0}{x} - \frac{c_1}{x+1} + \frac{c_2}{x+2} - \dots \\ &= (1 - c_1E + c_2E^2 - \dots)\, x^{(-1)} \\ &= (1-E)^n\, x^{(-1)} \\ &= (-)^n \Delta^n x^{(-1)} \\ &= \frac{n!}{x(x+1)\dots(x+n)}. \end{aligned}$$

Compare this solution with that given in text-books on algebra (*e.g.* C. Smith, *Algebra*, p. 307) and the advantage lies with the finite difference method. Moreover, the proof in the text-book is really no more than a verification and does not indicate how the result might have been, or in fact was, obtained.

Again, $$c_0 a - c_1(a-1) + c_2(a-2) - \dots = (1-E^{-1})^n a$$
$$= \Delta^n E^{-n} a = \Delta^n (a-n) = 0,$$

which is evidently superior to the proof by algebra (compare Smith, p. 308).

9. Further examples of algebraic identities are given which bring out clearly the method of approach. It is not to be expected that every such problem will be susceptible of treatment by operators, and it is often impracticable to use these methods. In many types of question, however, the application of separation of symbols is of distinct advantage.

*Example* 8. Prove that

$$\frac{1}{n}+\frac{x}{n(n+1)}+\frac{x^2}{n(n+1)(n+2)}+\ldots$$
$$=e^x\left\{\frac{1}{n}-\frac{x}{1!(n+1)}+\frac{x^2}{2!(n+2)}-\ldots\right\}.$$

The expression on the left

$$=n^{(-1)}+xn^{(-2)}+x^2n^{(-3)}+\ldots$$
$$=\left(1-\frac{x}{1!}\Delta+\frac{x^2}{2!}\Delta^2-\ldots\right)n^{(-1)}$$
$$=e^{-x\Delta}n^{(-1)}$$
$$=e^{x(1-E)}n^{(-1)}$$
$$=e^xe^{-xE}n^{(-1)}$$
$$=e^x\left(1-\frac{x}{1!}E+\frac{x^2}{2!}E^2-\ldots\right)n^{(-1)}$$
$$=e^x\left\{\frac{1}{n}-\frac{x}{1!(n+1)}+\frac{x^2}{2!(n+2)}-\ldots\right\}.$$

*Example* 9. Prove that

$$\{a+(a+1)+(a+2)+\ldots(a+n)\}\{a^2+(a+1)+(a+2)+\ldots+(a+n)\}$$
$$=a^3+(a+1)^3+\ldots+(a+n)^3.$$

The expression on the left

$$=(1+E+E^2+\ldots E^n)a\{(1+E+E^2+\ldots+E^n)a+a^2-a\}$$
$$=\frac{E^{n+1}-1}{E-1}a\,.\left\{\frac{E^{n+1}-1}{E-1}a+(a^2-a)\right\}$$
$$=(E^{n+1}-1)\Delta^{-1}a\,.\,\{(E^{n+1}-1)\Delta^{-1}a+(a^2-a)\}$$
$$=(E^{n+1}-1)\{\tfrac{1}{2}(a^2-a)\}\{(E^{n+1}-1)\tfrac{1}{2}(a^2-a)+(a^2-a)\}$$
$$=\tfrac{1}{2}\{(a+n+1)^2-(a+n+1)-a^2+a\}\,.\,\tfrac{1}{2}\{(a+n+1)^2-(a+n+1)+a^2-a\}$$
$$=\tfrac{1}{4}[\{(a+n+1)^2-(a+n+1)\}^2-(a^2-a)^2]$$
$$=\tfrac{1}{4}(E^{n+1}-1)(a^2-a)^2$$
$$=\frac{1}{4}\frac{E^{n+1}-1}{E-1}\,.\,\Delta(a^4-2a^3+a^2)$$
$$=\frac{E^{n+1}-1}{E-1}a^3\quad\text{since }\Delta(a^4-2a^3+a^2)=4a^3$$
$$=(1+E+\ldots+E^n)a^3$$
$$=a^3+(a+1)^3+\ldots+(a+n)^3.$$

*Note.*—This example is a good illustration of the use of operators in summing such series as

$$(a+k)^r+(a+2k)^r+\ldots+(a+nk)^r.$$

Even for small values of $r$ the ordinary algebraic method involves a considerable amount of calculation and re-arrangement of the terms. By finite differences the work is appreciably curtailed.

*Example* 10. Prove that

$$1\,.\,n(n+1)+n\,.\,(n-1)\,n+\frac{n(n+1)}{1\,.\,2}\,.\,(n-2)\,(n-1)+\ldots = 2\,\frac{(2n+1)!}{(n+2)!\,(n-1)!}$$

(C. Smith, *Algebra*, p. 357).

If $n_r$ denote $n(n-1)(n-2)\ldots(n-r+1)/r!$, it is easy to shew that $\Delta n_r = n_{r-1}$, so that $\Delta^{-1} n_{r-1} = n_r$; whence $\Delta^k n_r = n_{r-k}$ and $\Delta^{-k} n_{r-k} = n_r$.

$$\begin{aligned}
&1\,.\,n(n+1)+n\,.\,(n-1)\,n+\frac{n(n+1)}{1\,.\,2}\,.\,(n-2)\,(n-1)+\ldots \\
&\quad=\left[1+nE^{-1}+\frac{n(n+1)}{1\,.\,2}\,E^{-2}+\ldots\right] n(n+1) \\
&\quad=(1-E^{-1})^{-n}\,2(n+1)_2 \\
&\quad=2E^n(E-1)^{-n}(n+1)_2 \\
&\quad=2E^n\Delta^{-n}(n+1)_2 \\
&\quad=2E^n(n+1)_{n+2} \\
&\quad=2(2n+1)_{n+2} \\
&\quad=\frac{2(2n+1)!}{(n+2)!\,(n-1)!}.
\end{aligned}$$

H. F.

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**925.** The slightest force, when it is applied to assist and guide the natural descent of its object, operates with irresistible weight.—Gibbon, *Decline and Fall*, ch. xxv, with application to the pro-Christian policy of the Emperor Jovian.

**926.** ON THROWING A CRICKET BALL. "The world's record for throwing a cricket ball is about 150 yards ... this means a velocity of projection of ... about 80 miles per hour .... The work done in throwing a cricket ball is not performed by the muscles of the arm, at any rate mainly, but is transmitted down the arm by a jerk produced by the sudden checking of the body and shoulder." (From an authoritative exposition of animal mechanics.)

How so? And how of a golf ball? Contrast the cracking of a whip lash: *cp.* Routh's *Advanced Dynamics*, § 583.

[If the hand can thus be swung from the shoulder at 80 miles per hour, the head of a golf stick can perhaps be swung at nearly twice that speed on account of the doubled length of leverage; which is communicated almost entirely to the light golf ball, and would give a flight of as much as four times 150 yards if the direction of projection could be equally favourable.]

**927.** But supposing that the League stopped 50 per cent. of possible wars—and the percentage was infinitely higher—would it not be worth while?—Report in the *Times*, 6th April, 1933, of a broadcast address on the League of Nations given by Sir Eric Drummond. [Per Mr. G. B. Ehrenborg.]

# DYNAMICS WITHOUT DIFFERENTIAL EQUATIONS.

By V. V. Narlikar.

This paper is devoted principally to the exposition of a new calculating process which may be useful for what are called "natural" dynamical problems. A few examples, mostly trivial, are worked out to illustrate the process. But it is possible that the method may be used with advantage for several important problems, and a paper concerning its use for problems of gravitational astronomy will be shortly published elsewhere.

The usual procedure is to transform a dynamical problem into one of differential equations, which are usually called "the equations of motion". The difficulties of the problem then mainly consist in solving a set of differential equations which are not always integrable, and which, even when integrable, may become unmanageable if suitable co-ordinates are not used. It is possible to avoid these and cognate difficulties of analysis if we look upon the equations of motion not as differential equations but as different aspects of the manner of evolution of the Hamiltonian system. To describe the latter it is sufficient to consider only one operator $D$ which may rightly be called the "Poisson Operator".

Let us consider a "natural" dynamical system characterized by the Hamiltonian function $H$. Let there be $n$ free co-ordinates, $q_1, q_2, \dots q_n$, the corresponding generalized momenta being denoted by $p_1, p_2, \dots p_n$. By the very definition of a "natural" system the time $t$ will affect $H$ only through the co-ordinates and the momenta. Hence the $2n$ equations of motion

$$\frac{\partial H}{\partial p_r}=\dot{q}_r, \quad \frac{\partial H}{\partial q_r}=-\dot{p}_r \qquad r=1, 2, \dots n$$

are connected by the integral of energy, $H=h$(constant). A dot is used to denote a differentiation with regard to $t$. Suppose for the sake of convenience, that the motion commences from a certain epoch, $t=t_0$, when

$$q_r=\alpha_r, \quad p_r=\beta_r. \qquad r=1, 2, \dots n.$$

From the quantitative point of view the dynamical motion may be said to be completely determined when every $q$ and $p$ can be expressed as

$$q=q(\alpha\beta t), \quad p=p(\alpha\beta t).$$

The Poisson operator $D$ is given by

$$\frac{d}{dt}-\frac{\partial}{\partial t}=\left(\frac{\partial H}{\partial p_1}\frac{\partial}{\partial q_1}+\frac{\partial H}{\partial p_2}\frac{\partial}{\partial q_2}+\dots+\frac{\partial H}{\partial p_n}\frac{\partial}{\partial q_n}-\frac{\partial H}{\partial q_1}\frac{\partial}{\partial p_1}-\dots-\frac{\partial H}{\partial q_n}\frac{\partial}{\partial p_n}\right)\equiv D.$$

Thus, given any arbitrary function, $f=f(pq)$, $\dot{f}=Df$. The $2n$ equations of motion are only particular cases of this relation derived by putting $q$ or $p$ for $f$. Evidently,

$$\ddot{f}=D^2f, \ \dddot{f}=D^3f, \text{ etc.}$$

Assuming therefore that the Taylor expansion is possible at $t=t_0$, we obtain

$$f=f_0+(Df)_0(t-t_0)+\frac{1}{2!}(D^2f)_0(t-t_0)^2+\dots,$$

where the suffix "0" indicates that the value corresponding to $t=t_0$ is to be taken. We may write the expansion for $f$ in the form $f=e^{(t-t_0)D_0}f_0$, so that

$$q_r=e^{(t-t_0)D_0}\alpha_r, \quad p_r=e^{(t-t_0)D_0}\beta_r. \qquad r=1, 2, \dots n.$$

This is clearly the general solution of the "natural" dynamical problem with which we started, obtained, apparently, without integrating a single differential equation; to make it complete we give a theorem (which may be generalized at the cost of simplicity) on the convergence of the series.

If $H$ is a holomorphic function of the co-ordinates and momenta for $|q_r-\alpha_r|<\bar{q}$, $|p_r-\beta_r|<\bar{p}$, $r=1, 2, \dots n$, the $2n$ series for $q$'s and $p$'s are convergent for

$$|t-t_0|\leqslant \text{Min.}\left\{\frac{\bar{q}}{(H_2)_{\text{Max.}}}, \quad \frac{\bar{p}}{(H_1)_{\text{Max.}}}\right\}$$

where $H_1$ stands for the positional gradient of $H$, and $H_2$ for the motional gradient, *i.e.*

$$H_1=\sqrt{\left\{\left(\frac{\partial H}{\partial q_1}\right)^2+\left(\frac{\partial H}{\partial q_2}\right)^2+\dots+\left(\frac{\partial H}{\partial q_m}\right)^2\right\}}$$

$$H_2=\sqrt{\left\{\left(\frac{\partial H}{\partial p_1}\right)^2+\left(\frac{\partial H}{\partial p_2}\right)^2+\dots+\left(\frac{\partial H}{\partial p_m}\right)^2\right\}}.$$

This theorem follows immediately from Cauchy's existence theorem as demonstrated by Picard (*Traité d'Analyse*, t. 2, chap. xi) and hence need not be proved here. It must be remarked, however, that the actual radius of convergence for each series may be greater than that given by this theorem.

A simple illustration for the analytical procedure given so far is presented by the one-dimensional harmonic oscillator for which we may take

$$H=\tfrac{1}{2}\dot{q}^2+\tfrac{1}{2}\mu q^2=\tfrac{1}{2}p^2+\tfrac{1}{2}\mu q^2; \quad \mu>0.$$

Here the operator $D=p\dfrac{\partial}{\partial q}-\mu q\dfrac{\partial}{\partial p}$.

Then $(q)_0=\alpha$, $(P)_0=\beta$, $(Dq)_0=\beta$, $(D^2q)_0=-\mu\alpha$, $(D^3q)_0=-\mu\beta$, etc.

Thus
$$q=\alpha\left[1-\mu\frac{(t-t_0)^2}{2!}+\mu^2\frac{(t-t_0)^4}{4!}-\dots\right]$$
$$+\frac{\beta}{\sqrt{\mu}}\left[\mu^{1/2}(t-t_0)-\mu^{3/2}\frac{(t-t_0)^3}{3!}+\dots\right]$$
$$=\alpha\cos\sqrt{\mu}(t-t_0)+\frac{\beta}{\sqrt{\mu}}\sin\sqrt{\mu}(t-t_0).$$

Similarly $p=-\alpha\sqrt{\mu}\sin\sqrt{\mu}(t-t_0)+\beta\cos\sqrt{\mu}(t-t_0)$.

We know that this solution holds good for all time. Let us determine, however, the least radius of convergence as given by our convergence theorem. Assume, for the sake of definiteness, $\alpha > \frac{\beta}{\sqrt{\mu}} \geqslant 0$.

Let $$|q-\alpha| < \bar{q} = (\sqrt{2}-1)\alpha, \quad |p-\beta| < \bar{p} = 2\alpha\sqrt{\mu}:$$

since $H_1 = \mu|q|$ and $H_2 = |p|$,

$$(H_1)_{\text{Max.}} < \sqrt{2}\alpha\mu, \quad (H_2)_{\text{Max.}} < 3\alpha\sqrt{\mu}.$$

Therefore $$|t-t_0| \leqslant \text{Min.} \left\{ \frac{\bar{q}}{(H_2)_{\text{Max.}}}, \frac{\bar{p}}{(H_1)_{\text{Max.}}} \right\}$$

$$\leqslant \frac{\sqrt{2}-1}{3\sqrt{\mu}}.$$

This is the " least " radius of convergence for the series for $q$ and $p$ as given by the convergence theorem.

We have so far confined ourselves to expansions in ascending powers of $t-t_0$. The work of computing the orbits is, however, much simplified if expansions in powers of $q-\alpha$ are obtained. We will proceed now to discuss the circumstances under which such expansions are possible.

It is well known that the Hamiltonian system of equations is the first Pfaff's system of equations corresponding to the differential form

$$\Phi_d \equiv p_1 dq_1 + p_2 dq_2 + \ldots + p_n dq_n - H\,dt.$$

This means that if any admissible variation of $p$, $q$, $t$ is denoted by the operator $\delta$, the equation

$$\delta\Phi_d - d\Phi_\delta = 0$$

resolves into the Hamiltonian set. As a result of this any two differential forms $\Phi_d$ and $\Psi_d$ will give the same Hamiltonian set if $\Phi_d - \Psi_d$ is a complete differential form in $p$, $q$, $t$ only.

Let us now suppose that it is desired to obtain expansions in ascending powers of $q_1 - \alpha_1$, assuming, of course, that such expansions are valid. Using the integral of energy we may write

$$\Phi_d = p_2 dq_2 + p_3 dq_3 + \ldots + p_n dq_n - H' dq_1 - d(ht),$$

where $-H'$ is a root of the energy equation when the latter is looked upon as an equation for $p_1$. It will be remembered that $t$ was assumed not to appear explicitly in $H$; the corresponding assumption when the independent variable is $q_1$ is that the latter does not appear explicitly in $H'$. From the relation between $H'$ and $H$ it is evident that $q_1$ will or will not affect $H'$ and $H$ simultaneously. In short, $q_1$ must be a cyclic co-ordinate for the system $H$. The operator $D$ is now replaced by

$$D' \equiv \left( \frac{\partial H'}{\partial p_2} \frac{\partial}{\partial q_2} + \ldots + \frac{\partial H'}{\partial p_n} \frac{\partial}{\partial q_n} - \frac{\partial H'}{\partial q_2} \frac{\partial}{\partial p_2} - \ldots - \frac{\partial H'}{\partial q_n} \frac{\partial}{\partial p_n} \right).$$

Then as before

$$q_r = e^{(q_1-\alpha_1)D_0'} \alpha_r, \quad p_r = e^{(q_1-\alpha_1)D_0'} \beta_r. \quad r = 2, 3, \ldots n.$$

The domain of convergence for $q_1$ may be obtained in the same manner as the domain for $t$. It must be remarked, however, that a co-ordinate, say, $q_2$, may be a multiple-valued function of $q_1$, while all the co-ordinates can be only single-valued functions of $t$. The difficulties arising from this source are in general special for each problem and a general discussion about them will not be attempted here.

As an illustration of the procedure explained in the last paragraph, consider the motion of a particle of mass $m$ which is attracted to a particle of unit mass according to the Newtonian law of gravitation. The latter particle may be taken to be fixed at the origin. It is customary to treat this problem in polar co-ordinates, but we shall stick to the $q$-$p$ notation by putting

$$q_1=r,\ q_2=\theta,\ q_3=\phi,\ p_1=m\dot{r},\ p_2=mr^2\dot{\theta},\ p_3=mr^2\sin^2\theta\dot{\phi}.$$

Then the integral of energy is

$$H\equiv\frac{1}{2m}\left[p_1^2+\frac{p_2^2}{q_1^2}+\frac{p_3^2}{q_1^2\sin^2 q_2}\right]-\frac{m\mu}{q_1}=h,$$

where $h$ stands for the gravitational constant. Since $q_3$ is a cyclic co-ordinate, we may proceed to obtain expansions in powers of $q_3-\alpha_3$ supposing that they exist. Solving the energy equation for $p_3$, we may take

$$-p_3=H'=-\sqrt{m}\,q_1\sin q_2 L,$$

where

$$L=\sqrt{\left\{2h+\frac{2m\mu}{q_1}-\frac{1}{m}\left(p_1^2+\frac{p_2^2}{q_1^2}\right)\right\}}.$$

The operator

$$D'\equiv\frac{p_1q_1\sin q_2}{\sqrt{m}L}\frac{\partial}{\partial q_1}+\frac{p_2\sin q_2}{\sqrt{m}Lq_1}\frac{\partial}{\partial q_2}+\sqrt{m}\,q_1\cos q_2\frac{\partial}{\partial p_2}$$
$$+\left(\sqrt{m}\sin q_2 L-m^{3/2}\frac{\mu\sin q_2}{q_1L}+\frac{p_2^2\sin q_2}{\sqrt{m}Lq_1^2}\right)\frac{\partial}{\partial p_1}.$$

For the required expansions it will be found convenient to work out in the beginning $D'L$, which is given by

$$D'L=-\frac{1}{\sqrt{m}}\left(\frac{p_2\cos q_2}{q_1}+p_1\sin q_2\right).$$

Also

$$D'^2L=-L+\frac{m\mu\sin^2 q_2}{q_1L},$$

*i.e.*

$$D'^2\frac{H'}{\sqrt{m}\,q_1\sin q_2}=-\frac{H'}{\sqrt{m}\,q_1\sin q_2}+\frac{m^{3/2}\mu\sin^3 q_2}{H'}.$$

Hence

$$D'^2\frac{1}{q_1\sin q_2}=-\frac{1}{q_1\sin q_2}+\frac{m^2\mu}{\beta_3^2}\sin^3 q_2,$$

$$D'^3\frac{1}{q_1\sin q_2}=-D'\frac{1}{q_1\sin q_2}+\frac{m^2\mu}{\beta_3^2}D'(\sin^3 q_2),\ \text{etc.}$$

Collecting terms, we obtain the following expansion in polar co-ordinates :

$$\frac{1}{r\sin\theta}=\frac{m^2\mu}{\beta_3^2}\sin^3\theta_0+\left\{\frac{1}{r_0\sin\theta_0}-\frac{m^2\mu}{\beta_3^2}\sin^3\theta_0\right\}\cos(\phi-\phi_0)$$
$$+\left\{\frac{\beta_1}{\beta_3}\sin\theta_0+\frac{\beta_2}{\beta_3}\frac{\cos\theta_0}{r_0}\right\}\sin(\phi-\phi_0)+b_1\frac{(\phi-\phi_0)^3}{3!}+b_2\frac{(\phi-\phi_0)^4}{4!}$$
$$+(b_3-b_1)\frac{(\phi-\phi_0)^5}{5!}+(b_4-b_2)\frac{(\phi-\phi_0)^6}{6!}+(b_5-b_3+b_1)\frac{(\phi-\phi_0)^7}{7!}+\ldots,$$

where $$b_r=\frac{m^2\mu}{\beta_3^2}[D'^r(\sin^3\theta)]_0. \quad r=0,\ 1,\ 2,\ \ldots .$$

The above expansion is probably new. In the particular case where the motion is supposed to be in the plane $\theta=\frac{1}{2}\pi$,

$$\beta_2=0 \quad \text{and} \quad b_r=0,\ r=1,\ 2,\ \ldots .$$

The expansion then reduces to the well-known form

$$\frac{1}{r}=\frac{m^2\mu}{\beta_3^2}+\left\{\frac{1}{r_0}-\frac{m^2\mu}{\beta_3^2}\right\}\cos(\phi-\phi_0)+\frac{\beta_1}{\beta_3}\sin(\phi-\phi_0).$$

The expansions for the co-ordinates themselves are not always happy but it is sometimes possible to discover functions of co-ordinates for which simple expansions exist. Such functions are either suggested by the peculiarities of the problem under investigation (just as $1/q_1\sin q_2$ was suggested by $D'^2L$) or by the first few operations on an arbitrary function of one or two co-ordinates. Suppose we want to discover such a function of $q_2$ if it exists. Let $f(q_2)$ be an arbitrary function, the results of the first two operations on which are as follows :

$$D'f=\frac{p_1\sin q_2}{\sqrt{mLq_1}}f',$$
$$D'^2f=\sin q_2\cos q_2f'+\frac{p_2^2\sin q_2}{mL^2q_1^2}\left[\frac{\partial}{\partial q_2}(\sin q_2f')+f'\cos q_2\right],$$

where $f'$ stands for $\partial f/\partial q_2$. These operations suggest the function $\cot q_2$. We have then

$$f(q_2)=\cot q_2, \quad D'f=-\frac{p_2\operatorname{cosec} q_2}{\sqrt{mLq_1}}, \quad D'^2f=-\cot q_2, \text{ etc.}$$

Hence, by collecting the terms of the expansion, we obtain

$$\cot q_2=\cot\alpha_2\cos(q_3-\alpha_3)+\frac{\beta_2}{\beta_3}\sin(q_3-\alpha_3).$$

This relation gives that the orbit is a plane, the inclination of which as determined by initial conditions being also given.

The method may be useful where the progress of the motion or the path is to be numerically calculated. The following two problems may be suggested to the reader : (1) the orbit of the spherical pendulum, and (2) the evaluation of the five constants of the general equation of the conic $ax^2+by^2+2hxy+2gx+2fy+c=0$, in terms of the $\alpha$'s and $\beta$'s for the two-body problem. V. V. N.

# ON THE GRAPHICAL DISCRIMINATION OF THE CUBIC AND OF THE QUARTIC.

By J. P. Dalton.

1. As a rule graphical methods are more tedious than an analytical treatment; but they are welcomed by the weaker student on account of the illumination which they shed upon relations that would otherwise remain somewhat obscure.

Two notes * which have recently appeared in the *Mathematical Gazette* dealing with the graphical discrimination of the cubic tempt me to publish methods of discussing the roots of the cubic and of the quartic which have been in use here for some time and have stood the test of experience.

The discrimination proceeds in two stages. (i) The shape of the curve representing the cubic or quartic function (with rectangular axes) settles the *possible* number of real roots. (ii) For each particular shape the *actual* number of real roots is determined by the position of the $x$-axis relative to the turning points of the curve. The former criterion is based upon a knowledge of the derivatives of the function, the latter upon the ordinates of the turning points in their relation to auxiliary parameters or to auxiliary loci.

As only the zeros of the function are in question no generality is lost by making unity the coefficient of the highest power of the independent variable. For the rest, the usual notation is here used for the associated functions, except that $\delta$ represents the discriminant of a cubic, while $\Delta$ is reserved for that of a quartic.

In tabulating categories it will be convenient to use the abbreviations: $PI$=point of inflection; $TP$=turning point; $R$=real; $C$=complex; $D$=distinct; $E$=equal.

2. The Cubic.

By moving the $y$-axis parallel to itself to pass through the unique, real point of inflection, a cubic function may be put in the standard form:

$$y=x^3+3Hx+G. \qquad \text{.....(1)}$$

This translation has no effect upon the reality or otherwise of the roots. Successive derivatives are

$$y'=3(x^2+H);\ y''=6x;\ y'''=6;$$

and give, as shape criteria,

| | Conditions. | Number of $RTP$. | Maximum possible number of $RD$ roots. |
|---|---|---|---|
| ($a$) | $H>0$ | None | One |
| ($b$) | $H=0$ | None | One |
| ($c$) | $H<0$ | Two | Three |

* A. Hinckley, *Math. Gazette*, XV, No. 214, p. 424 (1931).
H. E. Piggott, *Math. Gazette*, XVI, No. 218, p. 126 (1932).

The usual diagrams reveal the change of shape due to the progressive change in the value of $H$; they also render evident the fact that in cases ($a$) and ($b$) the cubic (1) has, in general, one and only one real root, opposite in sign to that of $G$. The exception is when $H=0=G$. The curve has then a horizontal tangent at its point of inflection, and that tangent is the $x$-axis. In that case the cubic has three coincident zero roots.

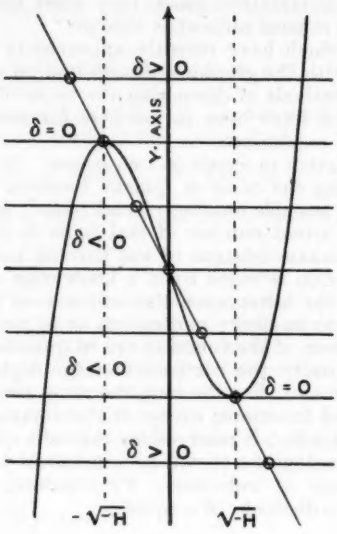


FIG. 1.

When $H<0$ (Fig. 1) the tangents at the turning points divide the plane into three regions which are such that when the $x$-axis lies in the central region the product of the ordinates of the turning points is negative, while that product is positive when the $x$-axis lies in either of the outer regions. On either boundary the product of the ordinates is zero.

Turning points of the cubic are given by (1) in conjunction with

$$x^2+H=0. \quad \text{.....(2)}$$

Hence,* at these points,

$$y^2-2Gy+G^2+4H^3=0, \quad \text{.....(3)}$$

and

$$y=2Hx+G. \quad \text{.....(4)}$$

From (3) the product of the ordinates of the turning points is

$$\delta=(G^2+4H^3).$$

* Burnside and Panton, *Theory of Equations*, I, 9th ed. p. 164 (1928).

But abscissae may be used instead of ordinates. The auxiliary locus (4) is the line joining the turning points ; it also passes through $(0, G)$ the point of inflection of the cubic, and it cuts the $x$-axis at $x = -G/2H$. If the $x$-axis pass between the turning points, this point of cut must lie within their $x$-range, whereas if the $x$-axis pass outside the $y$-range of the turning points, the point of cut must lie outside their $x$-range. From the diagram the following particulars are evident :

| Conditions. | Nature of Roots. |
|---|---|
| $-G/2H < -\sqrt{-H}$ | $2C$ $1RP$ |
| $-G/2H = -\sqrt{-H}$ | $2REN$ $1RP$ |
| $-\sqrt{-H} < -G/2H < 0$ | $2RDN$ $1RP$ |
| $G = 0$ | $1RN$ 1 zero $1RP$ |
| $0 < -G/2H < +\sqrt{-H}$ | $1RN$ $2RDP$ |
| $-G/2H = +\sqrt{-H}$ | $1RN$ $2REP$ |
| $-G/2H > +\sqrt{-H}$ | $1RN$ $2C$ |

When only the nature of the roots, irrespective of their sign, is in question, the various conditions condense into the usual statement in terms of the discriminant $\delta$.

3. THE QUARTIC.

By moving the $y$-axis parallel to itself to pass through the unique and real point where $y''' = 0$, a quartic reduces to the standard form :

$$y = x^4 + 6Hx^2 + 4Gx + (I - 3H^2), \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and this translation has no influence upon the reality or otherwise of the roots. Successive derivatives are :

$$y' = 4(x^3 + 3Hx + G) ; \; y'' = 12(x^2 + H) ; \; y''' = 24x ; \; y^{iv} = 24.$$

Hence,

| | Conditions. | Number of *RPI.* | Number of *RDTP.* | Maximum possible number of *RD* roots. |
|---|---|---|---|---|
| (*a*) | $H \geqslant 0$ | None | One | Two |
| (*b*) | $H < 0$ ; $\delta \geqslant 0$ | Two | One | Two |
| (*c*) | $H < 0$ ; $\delta < 0$ | Two | Three | Four |

Under (*b*) two cases are listed, for, although the function $y'$ has three real zeros when $\delta = 0$, two of these coincide ; the double root of $y'$ does not give a turning point of $y$, for the same value of $x$ makes $y'' = 0$, $y''' \neq 0$. Or, geometrically considered, the coalescing of two consecutive turning points results in a point of inflection with a horizontal tangent.

Turning points of the quartic are determined by (5) in conjunction with

$$x^3+3Hx+G=0. \quad \ldots\ldots(6)$$

Hence, at a turning point,*

$$(y-I)^3+27(Hy-J)^2=0. \quad \ldots\ldots(7)$$

The product of the ordinates of the turning points of the quartic is therefore

$$I^3-27J^2=\Delta=(I-3H^2)(I-12H^2)^2+27G^2(G^2+2J), \quad \ldots\ldots(8)$$

and this is the primary discriminating parameter.

$$(a) \quad H \geqslant 0.$$

The quartic cuts the $y$-axis at $y=I-3H^2$; there are no inflections, and the unique turning point is a minimum. The ordinate of the minimum point has the same sign as $\Delta$. Hence when $\Delta>0$, the $x$-axis passes below the minimum, and all the roots of the quartic are complex. If $\Delta=0$ the $x$-axis is tangent to the quartic at its lowest point; there are then two complex roots and a real double root, positive, zero, or negative, according as $G \gtrless 0$. If $\Delta<0$ the $x$-axis cuts the curve in two real distinct points whose abscissae are of the same sign (opposite to that of $G$) if $I>3H^2$, and of opposite signs if $I<3H^2$. The roots are also of opposite signs if $G=0$ and $\Delta<0$, but in that case (8) shows that $I-3H^2$ is necessarily negative, and no new condition is introduced.

The only exceptional case in this category arises when $\Delta=0$, $G=0$, $H=0$ (and so $\delta=0$, $I=0$). The quartic is then $y=x^4$ and has four coincident zero roots.

$$(b) \text{ and } (c) \quad H<0.$$

When $H<0$ there are two real points of inflection at $x=\pm\sqrt{-H}$, $y=I-8H^2 \pm 4G\sqrt{-H}$. The chord joining these points is the line

$$y=4Gx+I-8H^2. \quad \ldots\ldots(9)$$

Since, on the quartic,

$$y-4Gx-(I-12H^2)=(x^2+3H)^2, \quad \ldots\ldots(10)$$

the double tangent to the curve must be

$$y=4Gx+I-12H^2. \quad \ldots\ldots(11)$$

This tangent is parallel to the join of the inflections, parallel also to the tangent to the curve where it cuts the $y$-axis; and the segment between the points of contact is bisected by the $y$-axis. In fact, all inner and outer segments of chords with slope $4G$ are bisected by the $y$-axis.

Over the range $-\sqrt{-H}<x<\sqrt{-H}$, $x^2+H<0$, hence at points between its two inflections the curve must be convex upwards.

$$(b) \quad H<0\,;\ \delta \geqslant 0.$$

Subject to these conditions the quartic has but one turning point.

* Burnside and Panton, *loc. cit.*

Geometrically it is evident that this must be a minimum. The slope of the double tangent has the same sign as $G$. Fig. 2 illustrates the case $G<0$; reflection in the $y$-axis would correspond to $G>0$.

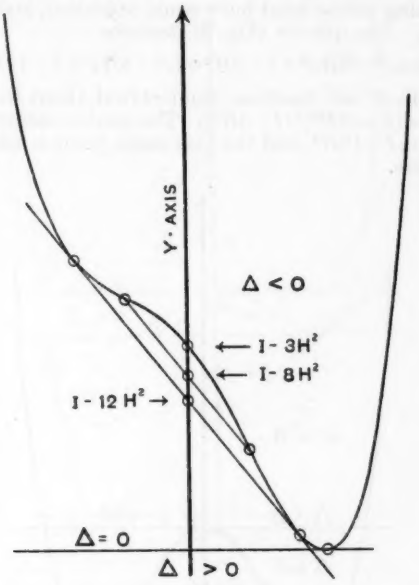


FIG. 2.

From the diagram it is clear that when the position of the $x$-axis is such that $\Delta>0$, the four roots of the quartic are complex. If $\Delta=0$, there are two complex roots, and two equal real roots of sign opposite to that of $G$. While if $\Delta<0$ two roots are complex, and two real and unequal of like sign (opposite to that of $G$) if $I>3H^2$, and of unlike signs if $I<3H^2$.

An exceptional case arises when $H<0$, $\delta=0$. One of the inflections has then a horizontal tangent at $(\pm\sqrt{-H}, I)$, where the sign of the ambiguity is that of $G$. If therefore the position of the $x$-axis is such that $I=0$ while $\delta$ also $=0$, the $x$-axis passes through the horizontal inflection, and the quartic has then three coincident real roots and one other. The sign of the triple root is the same as that of $G$ and opposite to that of the remaining root.

(c) $H<0$; $\delta<0$; $G=0$.

The discriminant of (7) is

$$\delta^*=3^6 . G^2 . \delta^3, \quad \text{.....(12)}$$

so that in general $\delta^*$ has the same sign as $\delta$. But it may happen that $G=0$, and then $\delta^*=0$ although $\delta \neq 0$. Let us first examine this special case.

Since $\delta \neq 0$ no turning points coincide; but, because $\delta^*=0$, two of these turning points must have equal ordinates, and these are the two minima. The quartic (Fig. 3) becomes

$$y=x^4+6Hx^2+I-3H^2=(x^2+3H)^2+I-12H^2, \quad \ldots\ldots\ldots(13)$$

a quadratic in $x^2$ and therefore symmetrical about the $y$-axis. We now have $\Delta=(I-12H^2)^2(I-3H^2)$. The double tangent is the horizontal line $y=I-12H^2$, and the maximum point is where the curve cuts the $y$-axis.

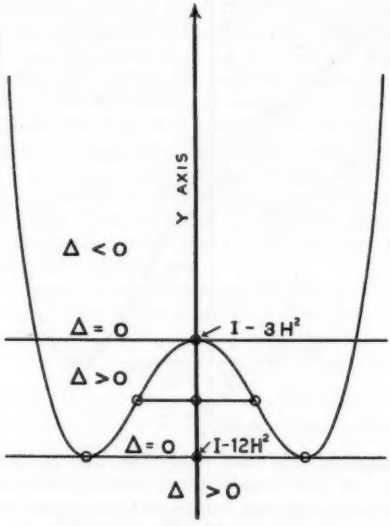


FIG. 3.

From the diagram we obtain:

| Conditions. | Nature of roots. |
|---|---|
| $\Delta>0$, $I>12H^2$ | $4C$ |
| $\Delta=0$, $I=12H^2$ | One pair $RN$; one pair $RP$ |
| $\Delta>0$, $3H^2<I<12H^2$ | $2RDN$ $2RDP$ |
| $\Delta=0$, $I=3H^2$ | $1RN$ 2 zero $1RP$ |
| $\Delta<0$, $I<3H^2$ | $1RN$ $1RP$ $2C$ |

(c) $H<0$; $\delta<0$; $G \neq 0$.

In the more general case, $G \neq 0$, and then $\delta^*<0$; hence the

ordinates of the turning points of the quartic are different as well as the abscissae. Fig. 4 illustrates a case in which $G>0$.

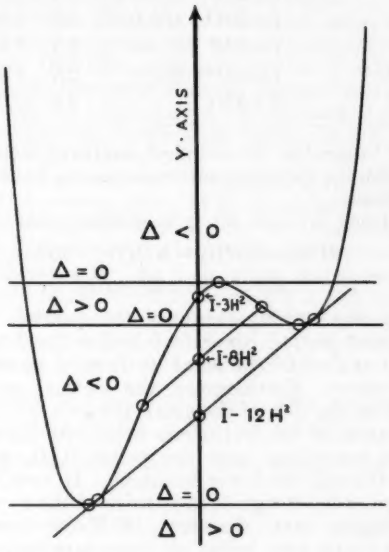


FIG. 4.

There are now four different regions such that when the $x$-axis lies within them in turn the product of the ordinates of the turning points, $\Delta$, is respectively $+, -, +, -$. The point where the double tangent (11) cuts the $y$-axis necessarily falls within the lower negative region of $\Delta$, for that point of cut must be below the upper minimum, and, when $I=12H^2$, we get, from (8), $\Delta=27G^2(16H^3-G^2)$ which is negative for all cases now under consideration.

Should the $x$-axis fall within either negative $\Delta$ region, the quartic has two complex and two real roots. For the upper region, that is to say, if $I-3H^2$ is also negative, the real roots are opposite in sign, while for the lower region the real roots are of the same sign, opposite to that of $G$.

If $I-12H^2>0$, and $\Delta>0$, the $x$-axis falls within the lower positive region, and the four roots of the quartic are then complex.

When the $x$-axis falls within the upper region which makes $\Delta>0$, $I-12H^2$ is then negative, and the four roots are real and distinct. Examination of the diagram, and of its mirror image, gives the further subdivisions :

| Conditions. | Nature of roots. |
|---|---|
| $\Delta>0$ ; $I<12H^2$ | All real |
| ,, ,, $I<3H^2$ ; $G>0$ | $1N$ $3P$ |
| ,, ,, $I<3H^2$ ; $G<0$ | $3N$ $1P$ |
| ,, ,, $I=3H^2$ ; $G>0$ | $1N$ 1 zero $2P$ |
| ,, ,, $I=3H^2$ ; $G<0$ | $2N$ 1 zero $1P$ |
| ,, ,, $I>3H^2$ | $2N$ $2P$ |

The double tangent is the simplest auxiliary locus to use in discriminating between the two positive $\Delta$ regions, but others also serve the same purpose.

From equations (5) and (6) it is evident that the rectangular hyperbola

$$H^2(x-G/H)(y-K/H)=-3G\delta, \quad \text{.....(14)}$$

where

$$K=J-2\delta=3J-2HI$$

passes through the turning points of the quartic. It also passes through the point $(0, I)$. Since $I=0$ makes $\Delta<0$, and the quartic cuts the $y$-axis at $I-3H^2$ the point $(0, I)$ must lie above the maximum of the curve. Furthermore, the vertical asymptote of the hyperbola lies on the side of the $y$-axis towards the lower minimum. Hence one branch of the hyperbola must pass through the upper minimum, the maximum, and the point $(0, I)$, while the other branch passes through the lower minimum. If, now, the $x$-axis pass through the upper $\Delta>0$ region, the centre of the hyperbola must lie below that region, and, therefore, $K/H<0$, hence $3J>2HI$; whereas if the $x$-axis pass below all three turning points, $K/H>0$, and $3J<2HI$. The boundary condition $3J=2HI$ requires the $x$-axis to lie in the lower negative $\Delta$ region, for we then have

$$H(I-12H^2)=3G^2,$$

and hence $I-12H^2<0$. Also, $I^3-27J^2=I^2(I-12H^2)$, thus $\Delta<0$.

This completes the discrimination of the roots of (5). The method is, indeed, somewhat laboured ; but it has two merits. It proceeds in an elementary geometrical fashion systematically to determine possible inflections and turning points, and it affords a precise geometrical interpretation of the discriminating parameters $\Delta$, $I-12H^2$ and $3J-2HI$.

J. P. D.

University of the Witwatersrand, Johannesburg, S.A.

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**928.** No more preposterous myth has ever been invented by Man than that of the alleged "Mystery of Woman". Woman is no more "mysterious" than are the Chinese or radio telephony or an equation in algebra, or Man himself. . . . A simple equation in algebra completely beats me and has ever done so, despite seven years' education (*sic*) at Winchester College and Oxford University. But algebra is not, therefore, a "mystery". Quite a number of folk understand it.—A. Corbett-Smith, *Woman—Theme and Variations*, 1932, pp. 17, 18. [Per Prof. E. H. Neville.]

## THE PILLORY.

THE following question was set in the University of London Intermediate B.Sc. Applied Mathematics paper, 1922.

"Two masses of 2 lbs. and 3 lbs. are fastened at the ends of a light string of length 2 feet and placed on a smooth horizontal shelf 5 feet above the ground. The 2 lb. mass is placed on the edge of the shelf and the 3 lb. mass one foot away from the edge on a line perpendicular to the edge through the mass of 2 lbs. If the mass is gently pushed over the edge find the time that elapses before the 3 lb. mass strikes the ground". ($g=32$ f.p.s.$^2$.)

The examiners' own answer is "·87 seconds if the edge of the shelf is rounded". The answers given in D. Humphrey's *Mechanics* are "·87 seconds if the edge is rounded; ·96 seconds if the edge is not rounded". The first answer can be obtained by assuming that both masses fall vertically after the 3 lb. mass leaves the shelf. The second answer can be obtained by assuming a jerk when the 3 lb. mass leaves the shelf and that subsequently the string is slack, the 3 lb. mass describing parabolic motion. Both sets of assumptions are definitely wrong. Consider the motion in detail.

1. The 2 lb. mass falls one foot freely. The velocity acquired is $\sqrt{2g}$ f.p.s., and the time taken $\sqrt{2/g}=\cdot 25$ seconds.

2. There is a jerk, after which both masses have a velocity $\frac{2}{5}\sqrt{2g}$ f.p.s.

3. During the next foot both masses have an acceleration $\frac{2}{5}g$ and acquire a velocity $V$ given by

$$V^2=\tfrac{4}{25}\cdot 2g+\frac{2\cdot 2g}{5}=\tfrac{28}{25}g,$$

whence $$V=5\cdot 987;$$

the time $t_1$ is given by

$$V=\tfrac{2}{5}\sqrt{2g}+\tfrac{3}{5}gt_1,$$

so that $t_1=\cdot 218$ seconds.

4. Just after the 3 lb. mass leaves the shelf the jerk gives the masses a vertical velocity $\frac{2}{5}V$; the 3 lb. mass having, in addition, a horizontal velocity $V$ (Fig. 1).

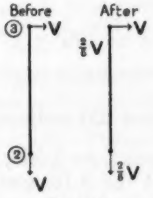


FIG. 1.

5. Motion until one mass reaches the ground. Take axes through the edge of the shelf (Fig. 2) and consider the parabolic motion of the centre of gravity.

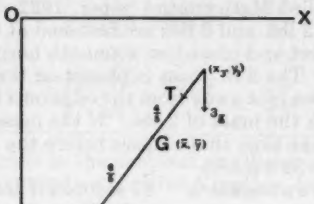


FIG. 2.

$$\dot{\bar{x}} = \text{constant} = \tfrac{3}{5}V,$$

and
$$\bar{x} = \tfrac{3}{5}Vt \text{ since } \bar{x}=0 \text{ when } t=0,$$

$$\bar{y} = \tfrac{4}{5} + \tfrac{3}{5}Vt + \tfrac{1}{2}gt^2 \quad (\bar{y} = \tfrac{4}{5} \text{ when } t=0),$$

$$x_3 = \bar{x} + \tfrac{4}{5}\sin\theta, \quad \text{.....(i)}$$

$$y_3 = \bar{y} - \tfrac{4}{5}\cos\theta, \quad \text{.....(ii)}$$

$$x_2 = \bar{x} - \tfrac{6}{5}\sin\theta, \quad \text{.....(iii)}$$

$$y_2 = \bar{y} + \tfrac{6}{5}\cos\theta. \quad \text{.....(iv)}$$

Now consider motion relative to the centre of gravity. We find

$$\ddot{\theta} = 0,$$

$$\dot{\theta} = \text{constant} = \tfrac{1}{2}V,$$

$$\theta = \tfrac{1}{2}Vt \text{ (since initially } \theta = t = 0).$$

Also
$$T = 3 \,.\, \tfrac{4}{5} \,.\, \dot{\theta}^2 = \tfrac{3}{5}V^2 > 0;$$

thus in this part of the motion the string can never become slack.

6. When $y_2 = 5$ we have the time $t_2$ for this part of the motion given by the equation

$$5 = \tfrac{4}{5} + \tfrac{3}{5}Vt_2 + \tfrac{1}{2}gt_2^2 + \tfrac{6}{5}\cos\tfrac{1}{2}Vt_2.$$

This can be solved by successive approximation giving $t_2 = \cdot 420$ seconds,

$$\theta = \tfrac{1}{2}Vt = 1{\cdot}257 \text{ radians} = 72°.$$

Then using equations (i)-(iv), $\bar{x} = 1{\cdot}509$, $\bar{y} = 4{\cdot}628$ : then $y_3 = 4{\cdot}381$ and is less than 5, so that the 3 lb. mass is still in the air. The motion after the impact of the 2 lb. mass depends on the coefficients

of elasticity and friction, so after all our work we find the problem insoluble with the data given.

The following questions of a similar nature are given in A. S. Ramsey's *Dynamics*.

"Two equal particles $A$ and $B$ are connected by a light inextensible string of length $a$, which is stretched at full length perpendicular to the edge of the table. The particle $A$ is drawn just over the edge of the table and is then released from rest in this position. Describe the nature of the subsequent motion and show that after $B$ leaves the table the centre of inertia of the two particles describes a parabola of latus rectum $\frac{1}{2}a$" (p. 133, Ex. 9).

The second question is similar to the first as regards initial conditions. It then says: "Prove that when the string is first horizontal the distance of $A$ from the vertical through the edge of the table is $\frac{1}{4}a(\pi - 2)$ and find the tension in the string" (p. 136, Ex. 22; the answer given on p. 137 is "Half the weight of either particle").

These questions can be solved very readily by the method used in the question considered first. S. RICHARDSON.

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**929.** . . . This young writer studied Rev. Wright's novel, *The Re-Creation of Brian Kent*, according to the laws of compound mathematical probability, and I summarize briefly:

The hero, a criminal fleeing from justice in Chicago, arrives in a village in the Ozarks, the home of "Auntie Sue, the silver-haired and golden-hearted re-creator". Estimating that there are three thousand villages to which he might have fled, we have an initial probability of one in three thousand. The hero, drunk, drifts upon a roaring river, and it would take a hydrographic chart to determine the chances of his boat stopping on a certain sand-bar; but we figure conservatively one chance in two hundred, which makes the cumulative probability one in six hundred thousand. Auntie Sue has sent some Brazilian banknotes to the Chicago bank which the hero has robbed, and as there are eight thousand banks in America, that is an easy item to figure. The notes arrived on the very day that the hero could steal them, which introduces yet another element of uncertainty.

It is a very long novel, and there enter such elements as Auntie Sue's happening to select just the right one out of thirty thousand stenographers in the United States, to come and type the hero's manuscript; also the chance of the hero's faithless wife and her paramour selecting a cottage just across the river for a summer resort. With such striking coincidences the odds mount up fast, and when we get to the end we find the chances of this particular wish-fulfilment of a Christian (Disciples) clergyman ever being brought about by a law-abiding Providence are one in 3456 followed by thirty-two ciphers; or if you find it easier to say, one chance in three hundred and forty-five billion and six hundred millions of thousands of millions of millions of billions.—Upton Sinclair, *Money Writes*, ch. xviii.

**930.** LORD PALMERSTON AT EDINBURGH. To this seat of learning the Palmerstons next exiled their son and lodged him with Professor Stewart. There, for four hundred pounds a year, he enjoyed the conversation of a mathematical philosopher and his agreeable wife.—Philip Guedalla, *Palmerston* (1926), p. 42. [Dugald Stewart (1753-1828) taught mathematics at Edinburgh, but is chiefly remembered as a philosopher.]

## MATHEMATICAL NOTES.

**1075.** *Continued Fractions and Series.*

If in the general continued fraction, which there is no restriction in taking in the form

$$\frac{b_1}{a_1-}\ \frac{b_2}{a_2-}\ \frac{b_3}{a_3-}\ \cdots,$$

the denominator $q_n$ is unity for all values of $n$, then the recurrence formula for $p_n$ becomes a formula for the convergent $p_n/q_n$ itself. Since $q_n = a_n q_{n-1} - b_n q_{n-2}$, a necessary condition for $q_n$ to be unity for all values of $n$ is $1 = a_n - b_n$, that is, $a_n = 1 + b_n$, for $n \geqslant 3$. Conversely, if this condition is satisfied,

$$q_n - q_{n-1} = b_n(q_{n-1} - q_{n-2})$$
$$= b_{n-1}b_n(q_{n-2} - q_{n-3}) = \ldots = b_3 b_4 \ldots b_n(q_2 - q_1),$$

so that if $q_2$ is equal to $q_1$, all the $q_n$ have the same value, which is unity if $q_1 = 1$; since $q_1 = a_1$, $q_2 = a_1 a_2 - b_2$, the initial conditions are that $a_1$ is unity and that $a_2$, $b_2$ satisfy the same relation as the later $a_n$, $b_n$.

Further, since the recurrence relation is the same for $p_n$ as for $q_n$, $p_n - p_{n-1} = b_2 b_4 \ldots b_n(p_2 - p_1)$, and substituting $p_1 = b_1$, $p_2 = a_2 b_1 = b_1 + b_1 b_2$, we have $p_n - p_{n-1} = b_1 b_2 \ldots b_n$, whence by addition,

$$p_n = b_1 + b_1 b_2 + b_1 b_2 b_3 + \ldots + b_1 b_2 \ldots b_n.$$

Thus we have Euler's transformation:

*The nth convergent to the fraction*

$$\frac{b_1}{1-}\ \frac{b_2}{1+b_2-}\ \frac{b_3}{1+b_3-}\ \cdots$$

*is the sum to n terms of the series*

$$b_1 + b_1 b_2 + b_1 b_2 b_3 + \ldots.$$

E. H. N.

**1076.** *A Bibliographical Note.*

A. M. Legendre:

*Exercices de Calcul Intégral* ....
3 vols., 1811-1813, 1814-1817, 1816-1818.

*Traité des Fonctions Elliptiques* ....
3 vols., 1825, 1826, 1828-1832.

The title pages of the *Exercices*, on which, by the way, the author's name has the form Le Gendre, are dated 1811, 1817, 1816, and the set is catalogued sometimes as 1811-1817, sometimes, by reference to the first and last volumes only, as 1811-1816. The fact is that only part of the third volume appeared in 1816; pp. 173-175 are occupied by a summary, dated 1818, of the *Suite* which concludes the volume and the work, and since the *Table des Matières* is not at

the end but is followed by an *Addition* of 26 pages, it is easy to overlook the *Table* also and to miss the hidden preface for that reason.

To give the dates of the three volumes as 1811, 1817, 1818 is to make a very imperfect correction. The first volume, as issued in 1811, consists of three *Parties*, but a *Supplément* issued in 1813 without title page is usually bound into the volume although there is no mention of it on the title page or in the *Table des Matières*. The second volume also consists of three *Parties*, but these appeared separately, the first two, without title pages, in 1814 and 1815. Then came the beginning of the third volume, with the title page dated 1816, followed in 1817 by the last part of the second volume and a title page for that volume. The last part of the third volume was the last section of the work to be issued.

Comparing the title pages of the second and third volumes of the *Exercices* we might suppose that Legendre had decided as we should ourselves, that the best date for the title page of a composite volume is that of the section which completes it. But the sequel does not support this assumption. The *Exercices* consists in the main of investigations on elliptic integrals and Eulerian integrals ; the comparative formlessness which made it easy for the author to modify his methods and to introduce improvements as he devised them, rendered the work unsatisfying as a treatise, and the *Traité* was designed to be in effect a revised edition of the *Exercices*, omitting a small amount of material irrelevant to the principal topics and presenting the whole theory in a systematic and finished form. The work so planned was completed in the first two volumes of the *Traité*, which are dated 1825, 1826.

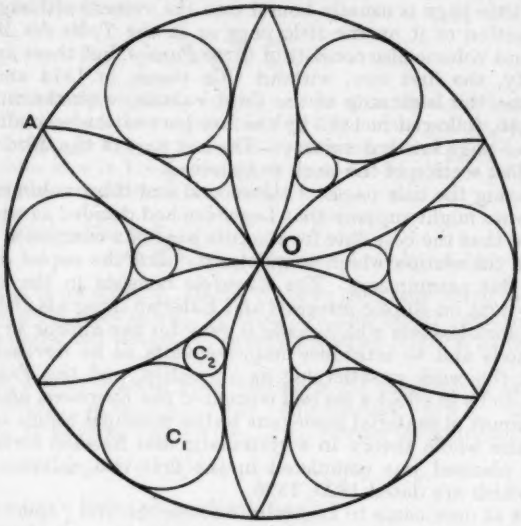
Almost at once came to Legendre's notice the first papers of Abel and Jacobi on the subject. Recognising their revolutionary importance, he proposed to rescue his own work from incompleteness by a volume of supplements to give an account of the new discoveries. In the preface to the *Premier Supplément*, dated 1828, in order to emphasise generously that Jacobi could not have derived even a hint from the section of the *Traité* which led most directly towards his work, Legendre reveals that the volume dated 1826 was not actually in the hands of readers until January of the following year—another discrepancy, but one that hardly calls for comment, since the possibility of ambiguity to this extent is unavoidable. What we have to notice however is that the third volume of the *Traité*, like the third volume of the *Exercices*, has its title page dated from its first sheets. The volume consists of three distinct *Suppléments*, of which the second is dated 1829 and the third 1832 ; if the volume is to have a single date, this must be four years later than the one invariably given.

E. H. N.

1077. *A " harmonic " set of circles.*

For me, and, I hope, for a great many teachers of geometry, there is a great attraction in a rider which allows of a neat proof and at the same time demands a figure which the pupils can be instructed to draw " beautifully ".

That shown in the figure here seems to me a good example. One of the best known patterns based on the hexagon inscribed in a circle is first drawn. Then a series of circles is to be inscribed as shown (the first two being drawn in the figure).




The results to be proved are :

(i) the distances from the centre of the original circle of the furthest points of the smaller circles are $r$, $\frac{1}{2}r$, $\frac{1}{3}r$, $\frac{1}{4}r$, $\frac{1}{5}r$, etc. ;

(ii) the radii of these circles are $\frac{r}{4}$, $\frac{r}{12}$, $\frac{r}{24}$*, and in general $\frac{r}{2n(n+1)}$.

To prove these, notice that the triangles $AOC_1$, $AOC_2$, ..., are right-angled at $O$, so that we get

$$\begin{aligned} r^2 &= (r+r_1)^2 - (r-r_1)^2 \\ &= (r+r_2)^2 - (\tfrac{1}{2}r-r_2)^2 \\ &= (r+r_3)^2 - (\tfrac{1}{2}r-r_3)^2, \end{aligned}$$

and so on, having between each stage and the next proved the corresponding result of (i) by using the suitable special case of the result

$$\sum_1^n \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

* The first two stages were pointed out to me by my pupils Messrs. G. S. Tarr and J. B. Frith, and the figure drawn by Mr. J. W. P. Scott.

In the general case,

$$r^2=(r+r_n)^2-\left(\frac{r}{n}-r_n\right)^2,$$

we get
$$\frac{r^2}{n^2}=rr_n\left\{2+\frac{2}{n}\right\}$$

and therefore
$$r_n=\frac{r}{2n(n+1)}.$$

With more advanced pupils it is interesting to *invert* with respect to the original circle. The inverse figure consists of sets of equal circles, touching pairs of parallel lines at a distance $r$ apart, and provides an easy proof of the properties stated above.

C. O. TUCKEY.

1078. *A problem in maxima and minima and a statical application.*

1. The problem is : to find a point $P$ within a triangle $ABC$, such that $h\,.\,AP+k\,.\,BP+l\,.\,CP$ is a minimum, where $h:k:l$ are given ratios.

If such a point exists within the triangle, $h$, $k$, $l$ must be such that any two are together greater than the third. .....(i)

For if, say, $k+l\leqslant h$, since $BP+PA>BA$, $CP+PA>CA$,

$$h\,.\,AP+k\,.\,BP+l\,.\,CP+(k+l-h)\,.\,AP>k\,.\,BA+l\,.\,CA$$

hence
$$h\,.\,AP+k\,.\,BP+l\,.\,CP>k\,.\,BA+l\,.\,CA;$$

accordingly the expression has its minimum value when $P$ is at $A$. Similarly $P$ is at $B$ or $C$ respectively if $l+h\leqslant k$ or $h+k\leqslant l$.

The following is an easy geometrical method of determining the position of $P$ when the condition (i) is satisfied.

On the sides $BC$, $CA$, $AB$ describe externally similar triangles $BCD$, $CAE$, $ABF$, so that

$$BC:CD:DB=EC:CA:AE=BF:FA:AB=h:k:l.$$

Then if $AD$ cuts the circle $BCD$ at $P$ it is easily seen that the circles $CAE$ and $ABF$ also pass through $P$, and that $B$, $P$, $E$ and $C$, $P$, $F$ are collinear. By Ptolemy's theorem

$$\frac{k}{h}a\,.\,BP+\frac{l}{h}a\,.\,CP=a\,.\,PD,$$

hence $h\,.\,AP+k\,.\,BP+l\,.\,CP=h(AP+PD)=h\,.\,AD$.

For any other point $P'$,

$$\frac{k}{h}a\,.\,BP'+\frac{l}{h}a\,.\,CP'\geqslant a\,.\,DP',$$

and so $h\,.\,AP'+k\,.\,BP'+l\,.\,CP'\geqslant h(DP'+AP')$

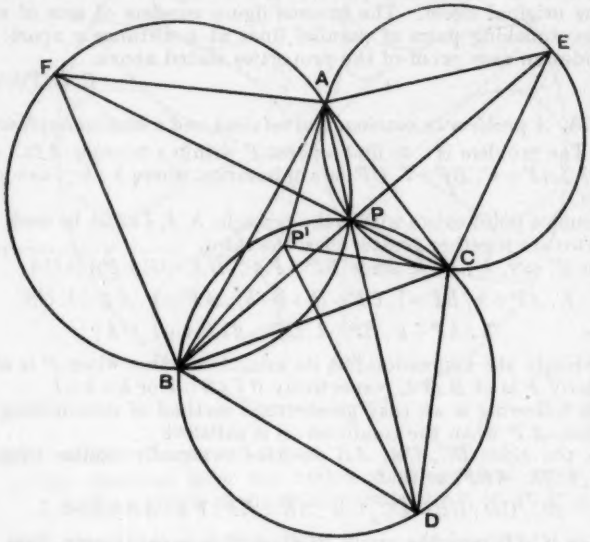
$$\geqslant h\,.\,AD$$

$$i.e.\quad \geqslant h\,.\,AP+k\,.\,BP+l\,.\,CP.$$

It follows that the minimum position is such that

$$\sin BPC:\sin CPA:\sin APB=h:k:l.$$

The late R. F. Davis gave a very neat statical discussion of this problem (*Math. Gazette*, XII, July 1924, p. 172, Note 726) by considering three inextensible strings knotted at $P$, passing through holes $A$, $B$, $C$ in a horizontal table, and carrying at their other ends weights proportional to $h$, $k$, and $l$. The position of stable equilibrium of the knot was deduced by the principle of minimum potential energy; but he omitted to mention that $h$, $k$ and $l$ must be subject to the condition (i). If this is not satisfied the knot moves to one or other of the vertices of the triangle.




2. Now for the statical application.

Suppose a lamina, resting in equilibrium on three rough supports $A, B, C$ in the same horizontal plane, to be acted upon by a gradually increasing horizontal couple in the plane of the lamina. Further suppose that equilibrium is about to be broken by the lamina turning about some point $P$ within the triangle. The pressures on the supports are $\Delta_1 W/\Delta$, $\Delta_2 W/\Delta$, $\Delta_3 W/\Delta$, where $W$ is the weight of the lamina, $\Delta$ the area of the triangle $ABC$, $\Delta_1$, $\Delta_2$, $\Delta_3$ those of the triangles $GBC$, $GCA$, $GAB$ where $G$ is the centre of gravity of the lamina. The frictions $F_1$, $F_2$, $F_3$ at $A$, $B$, $C$ being each limiting are respectively $\mu\Delta_1 W/\Delta$, etc., and act at right angles to $AP$, $BP$, $CP$ respectively and in the same sense.

Routh shows (*Analytical Statics*, I, § 185) that the position of the instantaneous centre of rotation is found by making the moment about $P$, that is, $F_1 . AP + F_2 . BP + F_3 . CP$ a minimum.

Hence by the preceding such a position of $P$ within the triangle exists if $F_1, F_2, F_3$ are proportional to the sides of a triangle, and this position is given by the relation

$$\sin BPC : \sin CPA : \sin APB = \Delta_1 : \Delta_2 : \Delta_3.$$

The fulfilment of this condition is achieved if $G$ lies within the triangle formed by joining the middle points of the sides of $ABC$. If this condition is not fulfilled the lamina will begin to turn about one or other of the points $A, B, C$. E. H. SMART.

1079. *The extended Leibnitz formula.*

Attention has lately been drawn to the formula

$$\frac{1}{f(D)}x^m T = x^m \frac{1}{f(D)} T + mx^{m-1}\left\{\frac{d}{dD}\frac{1}{f(D)}\right\} T + \dots \quad \dots\dots(1)$$

(Forsyth, *Differential Equations*, 1885, p. 63) and to the need of precautions in its application. The object of this note is to prove the formula in such a way that the precautions to be taken become obvious.

Let $n$ be the degree of the polynomial $f$. Suppose that the expansion of $\{f(a+b)\}^{-1}$ in ascending powers of $b$ is

$$\{f(a)\}^{-1} + g_1(a)\{f(a)\}^{-2}b + \dots + g_m(a)\{f(a)\}^{-m-1}b^m + \dots, \quad \dots\dots(2)$$

so that

$$g_m(a) = \frac{\{f(a)\}^{m+1}}{m!}\frac{d^m}{da^m}\frac{1}{f(a)},$$

a polynomial in $a$. Then there is an algebraic identity

$$f(a+b)[\{f(a)\}^m + g_1(a)\{f(a)\}^{m-1}b + \dots + g_m(a)b^m] = \{f(a)\}^{m+1} + h(a, b)b^{m+1}, \quad \dots\dots(3)$$

where $h(a, b)$ is a polynomial in $a, b$.

Here put $D$ for $a$, $D_1$ for $b$ and operate on a product $uv$, where $u$ is a function of $x$ only, $v$ a function of $x_1$ only, and $D = \partial/\partial x$, $D_1 = \partial/\partial x_1$. Afterwards put $x_1 = x$.

If $v = x_1^m$, and $\{f(D)\}^{m+1}u = T$, we have $D_1^{m+1}v = 0$ and

$$f(D)[x^m\{f(D)\}^m u + mx^{m-1}g_1(D)\{f(D)\}^{m-1}u + \dots + m!\,g_m(D)u] = x^m\{f(D)\}^{m+1}u = x^m T. \quad \dots\dots(4)$$

Thus a value of $\{f(D)\}^{-1}x^m T$ is

$$x^m\{f(D)\}^m u + mx^{m-1}g_1(D)\{f(D)\}^{m-1}u + \dots + m!\,g_m(D)u. \quad \dots\dots(5)$$

This is the formula to be proved since $\{f(D)\}^m u$ is a value of $\{f(D)\}^{-1}T$ and $g_r(D)\{f(D)\}^{m-r}u$ is a value of

$$\frac{1}{r!}\left\{\frac{d^r}{dD^r}\frac{1}{f(D)}\right\} T.$$

Thus to find a value of $\{f(D)\}^{-1}(x^m T)$ we are to form $u$ as

$$\{f(D)\}^{-m-1}T =$$

and then derive the terms of the expansion (5), from $u$ by direct operations. There is a complementary function, containing $(m+1)n$ arbitrary constants, in the most general value of $u$, but its contribution to the final result (5) must be reduced to zero by $f(D)$ on account of (4) and must therefore merge in the complementary function. Hence all the $(m+1)n$ constants may be taken to be zero if that is convenient.

*Or* the $(m+1)n$ constants may be introduced $n$ at a time by forming in succession the functions

$$\{f(D)\}^{-1}T,\ \{f(D)\}^{-2}T,\ \dots\ \{f(D)\}^{-m-1}T,$$

each from the one before it by the inverse operation $\{f(D)\}^{-1}$. Again each of the constants may be taken as zero.

*Example.* $$\frac{1}{D^2+1}(xe^{ix})=x\cdot\frac{1}{D^2+1}e^{ix}-\frac{2D}{(D^2+1)^2}e^{ix}.$$

Here we *may* take

$$\frac{1}{D^2+1}e^{ix}=\frac{x}{2i}e^{ix},$$

and then we *must* take

$$\frac{1}{(D^2+1)^2}e^{ix}\text{ to be }\frac{1}{(D^2+1)}\left(\frac{x}{2i}e^{ix}\right),$$

that is

$$-\tfrac{1}{8}(x^2+ix)e^{ix}+Ae^{ix}+Be^{-ix},$$

where $A$, $B$ may be zero. Then the value given for $\dfrac{1}{(D^2+1)}xe^{ix}$ is

$$\tfrac{1}{8}e^{ix}(-ix^2+x-8Ai+i)+2Bie^{-ix}.$$

If $f(D)$ is a simple power of a binomial, say $(D-\alpha)^q$,

$$\{f(D)\}^{-1}u=(D-\alpha)^{-q}u=e^{\alpha x}D^{-q}(e^{-\alpha x}u),$$

and the rule is fulfilled by our taking $x$ to vanish at the lower limit in each of the integrations indicated, *e.g.*, by taking $D^{-s}1$ to be $x^s/s!$.

Generally, if $\{f(D)\}^{-1}$ is reduced to partial fractions, $\Sigma A(D-\alpha)^{-q}$, each of the partial fractions may be taken by itself in virtue of the identity

$$f(a)\Sigma A(a-\alpha)^{-q}=1,$$

where we may put $D$ for $a$ and operate on $x^mT$. This is the rule proposed by F. Underwood (*Gazette*, May 1930).

1080. *A Method of Integration.*

The method of integration now proposed, while retaining the principle of logarithmic integration, avoids the explicit use of logarithms and brings out the result as an algebraic function.

If $$y=\frac{s}{t},\ \frac{1}{y}\frac{dy}{dx}=\frac{1}{s}\frac{ds}{dx}-\frac{1}{t}\frac{dt}{dx},$$

or $$\frac{dy}{dx}=y\left(t\frac{ds}{dt}-s\frac{dt}{dx}\right)\Big/ st:$$

thus
$$y=\frac{dy}{dx}\left(\frac{st}{t\dfrac{ds}{dx}-s\dfrac{dt}{dx}}\right). \qquad \text{.....(i)}$$

The fraction in brackets we call the integrating multiplier.

Now
$$\frac{1}{s^m}\cdot\frac{d}{dx}(s^m)=\frac{m}{s}\frac{ds}{dx},$$

and
$$\left\{\frac{1}{s^m}\frac{d}{dx}(s^m)-\frac{1}{t^n}\frac{d}{dx}(t^n)\right\}=\left(mt\frac{ds}{dx}-ns\frac{dt}{dx}\right)\Big/st,$$

and the reciprocal
$$st\Big/\left\{mt\frac{ds}{dx}-ns\frac{dt}{dx}\right\}$$

is the integrating multiplier. Note that these indices are as they appear in $y$ and not as in $dy/dx$.

Now take a given case:
$$y=\int\frac{(x^2-1)\,dx}{(1-x+x^2)^{\frac{1}{2}}(1+x+x^2)^{\frac{1}{2}}}.$$

The denominator contains two factors of which the first powers are:
$$1-x+x^2 \quad \text{and} \quad 1+x+x^2.$$

Thus
$$y=\frac{(x^2-1)(1-x+x^2)(1+x+x^2)}{(1-x+x^2)^{\frac{1}{2}}(1+x+x^2)^{\frac{1}{2}}\left(mt\dfrac{ds}{dx}-ms\dfrac{dt}{dx}\right)}$$
$$=\frac{(1-x+x^2)^{\frac{1}{2}}(x^2-1)}{(1+x+x^2)^{\frac{1}{2}}\left(mt\dfrac{ds}{dx}-ns\dfrac{dt}{dx}\right)};$$

and since $(1-x+x^2)^{\frac{1}{2}}$ and $(1+x+x^2)^{\frac{1}{2}}$ are in the numerator and denominator respectively, therefore the index $m=n=\frac{1}{2}$. Now build up and simplify the denominator of the integrating multiplier:
$$\tfrac{1}{2}\{(1+x+x^2)(2x-1)-(1-x+x^2)(2x+1)\}=x^2-1.$$

So
$$y=\frac{(1-x+x^2)^{\frac{1}{2}}(x^2-1)}{(1+x+x^2)^{\frac{1}{2}}(x^2-1)}=\left\{\frac{1-x+x^2}{1+x+x^2}\right\}^{\frac{1}{2}}.$$

Another example solved in the same way is
$$y=\int\frac{2x\,dx}{(1+x^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}}=\left\{\frac{1+x^2}{1-x^2}\right\}^{\frac{1}{2}}.$$

Now take a case in which one term of the integral does not appear in the differential:
$$y=\int\frac{a^2-2x^2}{(a^2+x^2)^{\frac{1}{2}}}\,dx;$$

thus $$y=\frac{a^2-2x^2}{(a^2+x^2)^{\frac{1}{2}}}\cdot\frac{st}{\left(mt\frac{ds}{dx}-ns\frac{dt}{dx}\right)}.$$

Here $t=a^2+x^2$, but $s$ is not known. Substituting

$$y=\frac{(a^2-2x^2)\,s}{(a^2+x^2)^{\frac{3}{2}}\left\{m(a^2+x^2)\frac{ds}{dx}-\frac{3}{2}s\cdot 2x\right\}};$$

also $$a^2-2x^2=ma^2\frac{ds}{dx}+mx^2\frac{ds}{dx}-3xs,$$

and we may satisfy this identity if $s$ is a multiple of $x$, when by equating like terms, we have

$$m\frac{ds}{dx}=1 \quad\text{and}\quad -2x^2=x^2-3xs,$$

whence $$m=1,\ s=x.$$

Thus $$m(a^2+x^2)\frac{ds}{dx}-\tfrac{3}{2}s\cdot 2x=a^2-2x^2,$$

and $$y=x/\{a^2+x^2\}^{\frac{1}{2}}.$$

Another example, usually treated by trigonometrical substitution, is

$$\begin{aligned}y&=\int\frac{dx}{x^2(1+x^2)^{\frac{1}{2}}}\\&=\frac{1}{x^2(1+x^2)^{\frac{1}{2}}}\cdot\frac{x(1+x^2)}{\left(mt\frac{ds}{dx}-ns\frac{dt}{dx}\right)}\\&=\frac{(1+x^2)^{\frac{1}{2}}}{x}\cdot\frac{1}{\{\frac{1}{2}\,.\,2x\,.\,x-1\,.\,(1+x^2)\,.\,1\}}\\&=\frac{(1+x^2)^{\frac{1}{2}}}{x}\cdot\frac{1}{x^2-1-x^2}\\&=-(1+x^2)^{\frac{1}{2}}/x.\end{aligned}$$

The reader will find other examples to test this method. Like other methods it is not of universal application and will fail in some cases.

C. I. DAVIDSON.

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**931.** We define $dx$ and $dy$ as quantities whose ratio $dy:dx$ has the same value as the limiting value of the ratio $\Delta y:\Delta x$; so that, if $y=f(x)$, $dy:dx=f'(x):1$, or $dy=f'(x)\,.\,dx$. With this meaning $dx$ and $dy$ are called differentials.—L. Toft and A. D. D. McKay, *Practical Mathematics*, p. 182. [Per Mr. J. T. Combridge.]

**932.** LORD PALMERSTON DURING THE CRIMEAN WAR. He wrote on everything—on the utility of Euclid to aspiring diplomats. . . .—Philip Guedalla, *Palmerston* (1926), p. 377.

# REVIEWS.

**Gli Elementi d'Euclide e la critica antica e moderna.** Editi da FEDERIGO ENRIQUES, col concorso di diversi collaboratori. Libro X. Pp. 333. Lire 30. 1932. (Zanichelli, Bologna)

The book before us is the third volume of the Italian translation of Euclid's *Elements*, with historical and critical commentary, under the general editorship of Professor Federigo Enriques. No more fully qualified editor could be found than Professor Enriques, as all those will know who have seen other volumes edited by him, such as his *Elementi di Geometria* (in which he had the co-operation of Ugo Amaldi), his work *Per la storia della logica*, and the three large volumes entitled *Questioni riguardanti le matematiche elementari*.

The fundamental idea which suggested the preparation of this edition is stated in the preface to the first volume (1925) to have been the practice of the "Scuola di Magistero" under which the critical study of a subject is placed on a historical basis, having recourse to classical models and observing the variations which have been introduced, by degrees, in successive treatises and commentaries that have appeared from time to time. "And certainly in no field can this programme be better carried out than in that of elementary geometry, since the development of this subject through the centuries is completely dominated by the great work of Euclid, insomuch that its history is bound up with that of Euclidean criticism." The first volume of this edition (1925) contains Books I-IV and the second (1930) Books V-IX. Each volume has a short historical introduction, with historical and critical notes on each definition, postulate, "common notion", or proposition, as it comes along. The bibliography at the beginning of Volume I will show that no important authorities have been left out of account, while naturally the references to the great Italians (Leonardo of Pisa, Tartaglia, Cardano and the rest) are particularly full. The editors immediately responsible for the various Books are Federigo Enriques and Maria Teresa Zapelloni for Book I, Maria Teresa Zapelloni for Books II, V, VI and X, Adriana Enriques for Book III, and Guido Rietti for Books VII-IX, while the notes to Book X are due to Ruth Struik.

It is well known that, of all the Books of Euclid, Book X is the most perfect in form and finish. De Morgan said of it, "The book has a completeness which none of the others (not even the fifth) can boast of, and we could almost suspect that Euclid, having arranged his materials in his own mind and having completely elaborated the tenth book, wrote the preceding books after it and did not live to revise them thoroughly". The subject of Book X is that of irrationals, that is, different varieties of irrational straight lines considered in relation to particular straight lines assumed as rational. After definitions of the terms "commensurable" and "incommensurable" as applied to magnitudes in general, "commensurable in length", "commensurable in square only" and "incommensurable in square" as applied to straight lines, and "rational" and "irrational" as applied to straight lines and squares respectively, Euclid passes to the famous proposition X. 1, which is at the root of the "method of exhaustion", and is used as a lemma to XII. 2, to the effect that "Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half (or the half itself), and from that which is left a magnitude greater than its half (or the half itself), and if this process be repeated continually, there will (at length) be left some magnitude which will be less than the lesser magnitude set out". Prop. 2 states the test for the incommensurability of two magnitudes founded on the process of finding the greatest common measure; and further propositions lead up to the fundamental theorem of Theaetetus (Prop. 9) that squares on straight lines commensurable in length have the ratio of a square number to a square number, and conversely; while squares on straight lines incommensurable in

length have not the ratio of a square number to a square number, and conversely. The main part of the Book is concerned with the construction and classification of various compound irrationals coming under the general forms $\sqrt{p} \pm \sqrt{q}$ and $\sqrt{(\sqrt{p} \pm \sqrt{q})}$, where $p$, $q$ are commensurable but not in the ratio of a square number to a square number, and where also *one* of the two expressions $\sqrt{p}$, $\sqrt{q}$ may be replaced by a non-surd expression ($a$, say). Euclid finds the square root of $\sqrt{p} \pm \sqrt{q}$ in a form equivalent to

$$\sqrt{\{\tfrac{1}{2}\sqrt{p} + \tfrac{1}{2}\sqrt{(p-q)}\}} \pm \sqrt{\{\tfrac{1}{2}\sqrt{p} - \tfrac{1}{2}\sqrt{(p-q)}\}}.$$

Conversely, he proves that to square the compound irrational $\sqrt{p} \pm \sqrt{q}$ gives the compound irrational $(p+q) \pm 2\sqrt{pq}$. The irrationals $\sqrt{p} \pm \sqrt{q}$ are of twelve kinds according to the values of $p$ and $q$ (six kinds with a positive and six with a negative sign); and again there are twelve kinds of compound irrational coming under the form $\sqrt{(\sqrt{p} \pm \sqrt{q})}$. All of these latter irrationals are roots of equations of the form $x^4 \pm \alpha x^2 \pm \beta = 0$, while all the other set of twelve are roots of equations of the form $x^2 \pm \alpha x \pm \beta = 0$. Euclid distinguishes one further sort of irrational, which he calls a "medial straight line", corresponding to the expression $\sqrt{(\sqrt{\alpha} \cdot \sqrt{\beta})}$ or $\sqrt[4]{A}$, which is the positive root of an equation of the form $x^4 = A$, where $A$ is not a square.

He has propositions corresponding to other algebraical devices, proving, for instance, that

$$\frac{A}{\sqrt{p} \pm \sqrt{q}} = \frac{A(\sqrt{p} \mp \sqrt{q})}{p-q}.$$

The whole method of the book is purely geometrical, showing the remarkable extent to which the Greek mathematicians could do by pure geometry what we now do with so much greater ease by algebra. Different explanations of the object of Book X have been given. The best seems still to be that put forward by Zeuthen. Our algebraic notation enables the precise character of any root of a particular equation of one of the above forms to be shown at once; the Greeks, however, had no algebraic notation; hence, Zeuthen suggests, they seem to have thought it necessary to compile once for all a repertory, in geometrical guise, of the different characteristic forms of roots resulting from the solution of such equations in different cases. The actual working by Euclid is necessarily complicated; no wonder therefore that Simon Stevin wrote (about 1600) "La difficulté du dixiesme Livre d'Euclide est à plusieurs devenue en horreur, voire jusqu'à l'appeler la croix des mathématiciens, matière trop dure à digérer, et en la quelle n'aperçoivent aucune utilité". The reading of the book in the text before us is however much facilitated by the use of modern abbreviations; while the development and the gist of the whole is made easy to follow by the addition, in the notes to each proposition, of the algebraical equivalent set out in accordance with a convenient and uniform plan. Unfortunately there are a large number of misprints in the volume, more or less serious (I have counted more than a hundred); but, as a rule, they are easy to detect, and the reader has only to be on his guard.

An interesting note by M. T. Zapelloni at the end of the volume shows the keenness with which the early Italian algebraists studied Book X, especially in connection with efforts to solve the cubic equation. Leonardo of Pisa (about 1226), in considering the equation $x^3 + 2x^2 + 10x = 20$ set him for solution by Giovanni of Palermo, proved that none of the irrationals of Euclid X would satisfy it. Bombelli in the sixteenth century tried to transform the expression $\sqrt[3]{(\sqrt{n} \pm m)}$ into a Euclidean binomial of the form $\sqrt{v} \pm u$. This has suggested to E. Bortolotti that it was on such lines that Scipione dal Ferro succeeded for the first time in solving the general cubic equation; and G. Vacca has followed up this idea. Suppose, he says, that

$$\sqrt[3]{(a + \sqrt{b})} + \sqrt[3]{(a - \sqrt{b})} = x.$$

Then by expanding $x^3$, it is easy to prove that

$$x^3 = 3x\sqrt{(a^2 - b)} + 2a.$$

Hence to obtain a root of the equation $x^3 = px + q$ (wanting the second term), we have only to substitute for $a$, $b$ in the above expression for $x$ the values

$$a = \tfrac{1}{2}q,\ b = \frac{q^2}{4} - \frac{p^3}{27},$$

which gives precisely the solution of Scipione dal Ferro. T. L. H.

**The Elements of Euclid.** Edited by ISAAC TODHUNTER. Introduction by SIR THOMAS HEATH. Pp. xviii, 298. 2s. 1933. Everyman's Library, 891. (Dent.)

No one, we imagine, now holds the view that children ought to begin their study of geometry by reading the *Elements*; we should be on guard, however, against too great a neglect of one of the world's greatest books. The aspiration of the present publishers "that this volume may easily be found as fascinating and as difficult to put down when once begun as any detective story" may be disappointed, but every teacher of geometry ought to be familiar with the contents of the *Elements* and this edition is both good and cheap. Considerations of format restricted the choice of the particular edition of Euclid to be reproduced in the Everyman series, but Simson's edition would have had claims even on an unrestricted choice; Todhunter substantially reproduced Simson's edition, adding some acute and illuminating notes. Thus we have here Books I-VI complete; Simson omitted Books X and XIII and Todhunter further omitted parts of Books XI and XII.

Sir Thomas Heath's introduction is concerned first with a plea for the reading of Euclid by students reasonably mature in geometrical intelligence and secondly with comments on Simson and Todhunter. All the relevant information is supplied without one unnecessary word.

We congratulate the publishers on their venture; the volume will hardly be a "best-seller"; on a narrow financial view it may not "pay". But a library which is to include the world's best books would not be complete without Euclid, for no other book has exercised on mathematical thought and development so wide, so lasting and so well-deserved an influence.

T. A. A. B.

**The Theory of Functions.** By E. C. TITCHMARSH. Pp. x, 454. 25s. 1932. (Oxford; at the Clarendon Press)

This work fills up a gap in English mathematical literature with a concise book on the theory of functions of a real and of a complex variable, introducing the student to modern problems and methods. A knowledge of elementary analysis only is required.

The book can be divided into three parts.

The first part (Chap. I) consists of parts of the theory of infinite series and integrals depending on the notions of uniform convergence. But the main parts of the book are the second (Chaps. II-IX) on the theory of functions of a complex variable and the third (Chaps. X-XIII) on functions of a real variable.

The second part gives a very wide account of the modern theory of functions of a complex variable. It is not the aim of the author to give a complete account of the subject in the present stage of its development (to do this would not be possible in a book of this size), but it is complete in the sense that it contains all the theorems that form the working tools of a modern analyst,

such as Jensen's theorem; the Vitali-Montel theorems on sequences of analytic functions; the Phragmén-Lindelöf theorems on the behaviour of functions in a sector, etc. But this is by no means the only purpose of the book. It gives an account of many branches of the subject in the modern spirit. The chapter on power series with a finite radius of convergence includes the Fatou theorem, gap theorems, the Abel theorem and some cases of Tauberian theorems. Chapters on integral functions and Dirichlet series give a good introduction to these subjects for a beginner.

The third part of the book is devoted to the Lebesgue theory of measure and integration, and to Fourier series. All that one needs to know in the theory of absolute integration for studying modern analysis can be found in the book. The chapter on Fourier series contains the usual tests of convergence, the Riesz-Fischer theorem, the uniqueness theorem, and a rapid discussion of Fourier integrals.

With a wide scope of information the book combines a very clear style. It is not difficult to read and the argument is always given in a very clear way.

The book is a very valuable contribution to the literature of the subject and can be very strongly recommended to the student both for the elementary and the advanced study of the subject. A. S. B.

**Plane Algebraic Curves.** By H. HILTON. Second edition. Pp. vii, 390. 28s. 1932. (Oxford University Press)

The main aims of this book continue to be the presentation of the forms and known properties of plane cubic and quartic curves of all possible types and the provision of abundant examples for practice.

Additional references to recent work are given in the new edition, and the number of examples has been increased so that there are now over 1500, many of which suggest ideas and results not dealt with in the text. It is rare to find a book of so advanced a nature with more than a sprinkling of examples, and these are usually either trivial or very difficult. Professor Hilton's profuse and varied collection and the excellent diagrams make his book specially interesting and useful to the student.

There is the one drawback that preoccupation with particular trees is apt to make one lose sight of the wood. One must not look to this book for general methods of obtaining canonical equations for curves, nor for any of the ideas following from the acceptance of the genus as the distinctive feature of an algebraic curve and the connection of these ideas with function theory. Function theory, indeed, is beyond the scope of the book, which makes little use of anything more complex than elementary algebra.

Genus, or deficiency as it is still called in the present book, remains throughout obscurely in the background. The vital parts it plays in correspondences and in the postulation of the variable intersections of a given curve with a curve of a linear system do not appear. An adjoint curve is not even defined in this edition. Although such questions belong to a general theory that is not indispensable for curves of order less than five, yet these curves, including as they do examples of genus 0, 1, 2 and 3, illustrate both the exceptional behaviours associated in the general theory with the first three values of the genus and the normal behaviour that is followed by all curves of genus three or more. The student who has mastered a good selection of Professor Hilton's examples and can adapt himself to the newer point of view will be thoroughly prepared to illustrate for himself the powerful theory which has been developed mainly in German and Italian schools since Salmon last revised his classical treatise. The treatises of Enriques, Severi, Severi-Löffler and Coolidge, published during the last dozen years, describe this theory in detail.

The book does far more than secure such a preparation, however. One must be thankful that the omission of modern insistence on the genus and on

the free use of higher dimensions has made available for use and reference such a rich mine of examples of the various combinations of singularities that may occur, so well-ordered a field of practice for acquiring agility in manipulating algebraic equations and their geometrical interpretation, as well as metrical results (such as those connected with foci) which have elegance and practical appeal that should not be forgotten.

The greater part of the second edition is a photographic reprint of the first, but considerable alterations have been made in the text of the chapters on Superlinear Branches, Quadratic Transformation, Use of Parameters, and Use of Parameter for Cubic Curves. These are difficult chapters in that they describe processes which use theories not fully developed in the book. The first is concerned with the method of obtaining the various expansions in the neighbourhood of a complicated singularity and its effect on Plücker's equations. This has been made more intelligible, making possible in turn a clearer description of the resolution of a complicated singularity by quadratic transformation.

Many new examples are given in the chapter on Use of Parameters, and the sections at the end explaining in general how to obtain a parametric representation for a rational or elliptic curve have been rewritten more clearly.

In the chapter on Intersections of Curves the important addition is made of Professor E. B. Elliott's proof of Noether's "$Af+B\phi$" theorem in the simplest case. This proof is particularly satisfying because it makes no use of counting of constants. Unfortunately the theorem regarding the postulation of an $r$-ic through certain of the intersections of an $n$-ic and an $N$-ic and its proof have been left in this chapter as Salmon left them, incomplete and misleading. It is to be regretted that the true theorem embracing all cases has not yet been made available in an English text-book, particularly as the essential result was known to Jacobi, namely, that "*There is a linear identity between the $(N+n-3)$-th powers of the left-hand sides of the equations of the $Nn$ points of intersection of an $n$-ic and an $N$-ic*". Elsewhere, in the proofs that the deficiency cannot be negative and that a curve of deficiency zero is rational, the independence of the conditions that certain points of a given curve should lie on another curve of given order has again been tacitly assumed.

The rewriting of the chapter on the application of elliptic functions to the non-singular cubic has simplified matters by giving the more important place to Weierstrass', instead of to Jacobi's elliptic functions, and by collecting at the beginning the particular cases of Abel's theorem that are the basis of applications to intersections of the cubic with lines, conics and other cubics.

There are only a few trivial misprints, and the photographic reproduction has been admirably done. The book in its second edition has been enriched both by addition and revision. It retains all its former good features, and remains a useful treasury of information and problems for those interested in algebraic curves.

F. Bath.

**Introduction to Higher Geometry.** By W. C. Graustein. Pp. xv, 486. 18s. 1930. (Macmillan)

Professor Graustein's object in this book is to provide the link which many students miss, between ordinary school "Analytical Geometry" or "Conics", and the more sophisticated points of view needed if geometry is to be studied to an advanced stage.

He accustoms the student to the use of coordinates with numerical suffixes, and to summation over these suffixes at appropriate moments. He introduces the idea of duality, and the use of coordinates for lines (in a plane) as well as for points. He takes the student in an easy and at the same time succinct manner over the elementary theory of projectivity, cross-ratio, Desargues' theorem, etc.; but no axiomatic development is attempted, and the Fundamental Theorem of Geometry appears only as an obvious corollary of theorems

on cross-ratios, proved earlier by elementary algebra; the existence of geometries in which this theorem does not hold, and whose coordinates are non-commutative, is not touched upon.

The relations between the projective, affine and metric groups are well brought out, and the projective definition of metrical properties is simply explained. Complex elements are treated simply as given by complex coordinates, and the definition in terms of involution is not given.

The geometry of conics is treated in a good deal of detail from all the points of view which have been introduced. It is perhaps unfortunate that the equation of a conic should sometimes be given in the unfamiliar and unsymmetric form

$$Ax_1^2+Bx_1x_2+Cx_2^2+Dx_1x_3+Ex_2x_3+Fx_3^2=0,$$

though

$$\Sigma a_{ij}x_ix_j=0$$

is also used.

The geometry of circles and of inversion in the plane is expounded, with some hint of the relation of this theory to that of one complex variable; the latter topic however is not treated very fully.

In the last chapter the author makes a few hasty excursions into three-dimensional geometry; elementary theory of quadrics, duality, etc. The relation between the geometry of circles in a plane and the projective geometry of a sphere, and finally, the elements of line geometry, using Plücker coordinates, are here sketched.

The book is properly named, and seems admirably adapted to its purpose. It may well be said that a student who has mastered it, though in a sense he will know little more than he has learned from the ordinary schoolbooks, will have undergone just that change of spirit which will enable him to begin the study of any branch of geometry in a serious way. PATRICK DU VAL.

**Projective Differential Geometry of Curves and Surfaces.** By E. P. LANE. Pp. xi, 321. 22s. 1932. (University of Chicago Press)

The classification of differential geometries in the spirit of Klein's Erlanger Programm has now reached the stage in which the various kinds have special treatises devoted to them; thus the second volume of Blaschke's *Vorlesungen über Differentialgeometrie* (Berlin, 1923) deals exclusively with affine geometry, while for projective geometry we have Fubini and Čech (2 vols., Bologna, 1926-7), and now this admirable account by Professor E. P. Lane of Chicago. There can no longer be any reason why English students of geometry should not realize the great progress that has been made in these matters during the last twenty years or so, in America, Germany and Italy.

The first systematic development of projective differential geometry, *i.e.* the study of differential properties of curves, etc., which are invariant under a general projective transformation, is due to Halphen, who in two memoirs, of 1878 and 1880, considered curves in the plane and in space from this point of view. Not a great deal more for curves has been discovered since Halphen's work, but his methods are not readily applicable to the study of surfaces, and it was not until the publication of Wilczynski's *Projective Differential Geometry of Curves and Ruled Surfaces* (Leipzig, 1906), that the subject really got going. Wilczynski's method, beginning with the parametric equations which express the homogeneous coordinates of a point (or other element) of the variety as functions of a certain number of parameters, is to find a completely integrable system of homogeneous linear partial differential equations of which the coordinates are a fundamental set of solutions. This system of equations then defines the variety except for a projective transformation, since the most general solution can be expressed as a linear combination, with constant coefficients, of the fundamental set. There are, however, further transforma-

tions which do not affect the configuration, namely, the multiplication of the homogeneous coordinates by a proportionality factor, and the introduction of new parameters in place of the old ones. The effect of these transformations on the differential equations must then be calculated, so as to obtain a complete system of geometrical invariants and covariants. The geometrical interpretation of these invariants follows. In a parallel treatment of the metrical geometry of surfaces, the starting-point would be Gauss's and Weingarten's formulae expressing the derivatives of the fundamental vectors $x_u$, $x_v$ and $\xi$ (unit vector along the normal) in terms of these vectors, the integrability conditions appearing as the formulae of Codazzi.

But just as Gauss began with the quadratic differential forms of the surface as fundamental, so is it possible to begin the projective differential theory on the basis of a system of differential forms, using Ricci's absolute differential calculus. This method was introduced by Fubini, and is developed in detail in the book mentioned earlier; Professor Lane employs Wilczynski's method for the most part, with modifications, and with continual reference to the work of the Italian school.

A particularly valuable feature of the book under review consists of the excellent summaries, not only at the beginning of each chapter but at the beginning of each section. The author explains clearly just what he is going to do and as a result makes the work much more readily intelligible. It is well written and well printed and eminently readable; in fact, it might well be taken as a model by other authors. F. P. W.

**1. Hydrodynamics.** By Sir Horace Lamb. Sixth Edition. Pp. xv, 738. 45s. 1932. (Cambridge University Press)

**2. Einführing in die Theorie der zähen Flüssigkeiten.** By W. Müller. Pp. x, 367. Rm. 22.80. 1932. (Akademische Verlagsgesellschaft, Leipzig)

1. It is fifty-four years ago since Sir Horace Lamb first published his *Treatise on the Mathematical Theory of the Motion of Fluids* as a smallish volume of some 250 pages. The second edition, published sixteen years afterwards, was more than twice the size of the first, and bore the altered title, *Hydrodynamics*. By the third edition, published in 1906, both the outward appearance and the general character of the book were determined. The classical theory of the motion of an ideal, inviscid, incompressible fluid, and the theory of waves, were being treated by a master. It is noteworthy that in each of the last three editions over 300 pages out of a total of some 700 have been devoted to waves. The excellent discussion of vortex motion has been given without explanation of the existence of vortices; one long chapter, of something well over a hundred pages, has been devoted to the motion of real, viscous fluids, incapable of slipping freely over their solid boundaries, and this chapter has included mention of some of the difficult questions of turbulent motion; whilst a final chapter has dealt with rotating masses of liquid.

In this, the sixth, edition the character of the book is, then, maintained. It remains the leading treatise on classical hydrodynamics, and serves as a corrective to some of the trends in recent compilations, such that, on the one hand, a report on hydrodynamics can be published without mention of tides or waves and very little mention of classical hydrodynamics at all, and, on the other hand, the theory of Oseen and the extensions of it made last year and the year before can be included in a section on classical hydrodynamics. Moreover, Sir Horace Lamb can rightly claim in his preface to the new edition that classical hydrodynamics has been found to have a widening field of practical applications. Nevertheless, these applications mostly require the inclusion of circulation and vorticity, for the explanation of whose existence recourse must be made to viscosity and boundary layers; and Sir Horace Lamb has nowhere included an adequate discussion of the production of circulation and vorticity in fluids of small viscosity. The account on pp. 106 and

107 of the flow of real fluids past projecting corners, which is intimately connected with this subject, is retained from previous editions. In this account the difference between the observed phenomena and those calculated on ideal fluid theory are attributed largely to cavitation; but there is now no doubt that although, in the case of water, for example, separation of dissolved air and evaporation would take place and prevent the establishment of the calculated infinite negative pressure, and therefore also the flow round the corner, yet, in fact, any viscosity, no matter how small, so modifies the motion, before any cavitation takes place, that no large negative pressures are ever brought into being. In fact Sir Horace Lamb himself says, "In actual liquids the problem is modified by viscosity, which prevents any slipping of the fluid immediately in contact with the tube, and must further exercise a considerable influence on such rapid differential motions of the fluid as are here in question". This, combined with some new remarks on p. 687 concerning separation of the boundary layer from curved surfaces, could easily be expanded into a complete explanation according to the boundary layer theory of the observed phenomena; and such an explanation could, indeed, have served so to join classical and modern hydrodynamics as to make them one continuous whole.

Something must also be said about the chapter on real fluids. Although within a single chapter, even with the valuable additions now made, it is no longer possible to give a completely adequate account of our present state of knowledge of viscous and turbulent flow, and there are several matters which one would certainly have liked to have seen more fully treated, yet it is a matter for admiration how much is included, and how clearly a difficult subject is presented.

In this edition, numerous new references are given, and the list of additions made includes the equations of motion in general orthogonal coordinates, electrical analogies, theorems and examples on the calculation of the forces and moments on an immersed cylinder in two-dimensional flow, a slight extension of the discussion of the Joukowski aerofoil, recent important theoretical work on tidal motion, waves of finite amplitude of permanent type, the influence of viscosity on sound waves of permanent type, the generation and maintainance of waves by wind, changes and additions to the discussion of atmospheric waves, a reference to the drag formula for a cylindrical obstacle with a Karman street in the wake, a reference to, and criticism of, the asymptotic theory of Oseen, a short discussion of dynamical similarity, the boundary layer equations for steady motion in two dimensions, and some discussion of the flow along a flat plate. One of the most important additions is, perhaps, the discussion of the influence of compressibility on the flow past an obstacle.

One or two misprints have survived from previous editions, and a few new ones have crept in—the omission of "to" at the top of p. 16, a reference on p. 17 to a deleted preface, $\frac{\partial p}{\partial x}$ for $\frac{\partial p}{\partial y}$ on p. 29, afforded for avoided, and different for differential on pp. 106, 107 (these are new), $c_1^2$ for $c_1$ and the omission of $\sigma$ on p. 313, $-w^2z$ for 0 on p. 697, and Goldstein, G. for Goldstein, S. on p. 731. The reference on p. 617 to two important papers by Prof. Filon is definitely misleading.

It remains only to add that the lucidity of style which characterized the previous editions is maintained in all the new matter added, and that this edition is as certain of success as its predecessors.

2. Since Herr Müller's book is devoted entirely to the study of the motion of viscous, incompressible fluids, omitting wave phenomena, he has been able to enlarge on exactly those matters which we would have liked to have seen more fully treated by Sir Horace Lamb, such as Prandtl's boundary layer theory, and the exact solutions of the equations of viscous motion, due to Hamel and Karman, in which the quadratic terms play a part. In the opening chapter, the equations of motion are found, and dynamical similarity, the dissipation

function, and transfer of momentum discussed (but not, strangely enough, the general equation for the circulation or the vorticity). The use of vector calculus then makes the transformation of the equations to polar coordinates, both cylindrical and spherical, a matter of not too great difficulty. There follow numerous examples of the solved and partly-solved problems of flow in pipes and channels, of the motion of solids in fluids, of the diffusion of vorticity, and the like, ending with a chapter on Oseen's asymptotic theory. Finally, a short discussion is given of the problems presented, and some of the results so far achieved, in the consideration of stability and turbulent motion. The character of the book is fundamentally mathematical, for although experimental results are sometimes given, and apparatus sketched, theoretical physics—the qualitative explanation of phenomena, calculable or not—is not much in evidence. There are, however, a number of excellent diagrams and photographs; and, in addition, the book is a good mathematical text-book of its kind. This having been said, it might be permissible to add that the book is certainly not as readable, nor the mathematics as simplified, as Sir Horace Lamb's (which is not surprising); and that some examples—the diffusion of vorticity, and two-dimensional slow motion, for instance—seem to the reviewer to have been too fully treated for a book of this character. Opportunities for simplified presentation of some of the mathematics of the originals have been missed, even when the simplification has been published, or is easily inferred from published works, such as in the cases of the Oseen formula for the drag of a sphere in slow motion, and G. I. Taylor's investigation of the stability of flow between rotating cylinders. Again I cannot forbear to mention that at least one-third of the quite considerable space devoted to motion varying with time could have been saved by the use of Heaviside's operational calculus, though I suppose it is too much to expect to see it used yet in a book like this. It must come in time, however. Finally, it would be pleasant to see again Taylor, G. I., and not Taylor, G. J., in a German book. It can be done—the *Handbuch der Experimental Physik* scores one out of three in its indexes. And, unlike some other German authors, Herr Müller gives Newton, I. quite correctly.

So as not to leave a false impression, let me repeat, however, that this is quite a good text-book of its kind. S. GOLDSTEIN.

**The Expanding Universe.** By SIR ARTHUR EDDINGTON. Pp. vii, 128. 3s. 6d. 1933. (Cambridge)

In 1917 de Sitter showed the existence of a solution of the gravitational field equations of general relativity corresponding to a universe in which all objects appear to have a velocity of recession from any given observer, depending on their distance from him. That this might describe the actual universe was subsequently demonstrated in a striking manner by the observations of Slipher and Hubble on the extra-galactic nebulae. Later it emerged from the theoretical investigations of Friedmann, Lemaître, Robertson, and others, that de Sitter's solution is best interpreted as a limiting form of one of a whole class of solutions corresponding to "expanding" universes. The particular member of this class favoured by Eddington gives one starting from unstable equilibrium as an "Einstein universe", with a radius and total mass depending on the cosmical constant. He conceives the initial equilibrium state as corresponding to a uniform distribution of matter, and the probable cause of departure from equilibrium as the formation of local condensations in the material. On this theory things began to happen in the universe some $10^{10}$ years ago.

Such in outline is the first part of the story told, and told brilliantly, by Sir Arthur Eddington. The scene is then changed, and he passes on to consider how the properties of the ultimate particles of matter, electrons and protons, are conditioned by their interaction with the rest of the universe. In his view,

"length" has a meaning only when measured against the radius of curvature of space at the point concerned. So he relates one term in the wave equation of the electron to this radius $R$, and also to the total number of particles $N$ in the universe. His theory of the expanding universe gives him another relation between these quantities, and the two relations together enable him to evaluate them in terms only of quantities measurable in the laboratory. In the language of a certain examinee in elementary relativity, he has now "solved the universe"! On the one hand he uses his laboratory values of $N$, $R$ to predict the rate of recession of the spiral nebulae, and gets remarkable agreement with the astronomical value. On the other hand he finds it possible to modify his equation relating $N$, $R$ to the mass of the electron to include also the mass of the proton. His resulting theoretical ratio of these masses is 1847·60—experimental value $1847{\cdot}0 \pm 0{\cdot}5$!

The tale is a fascinating one, and it all seems too good *not* to be true. But the tentative character of the work is frankly recognised throughout. One may be pardoned, however, for asking if it is not too tentative for presentation in this form to the general reader. Is it quite fair to offer him arguments whose validity he has not the means to judge, before they have had adequate examination and support from the general body of those who are qualified to express an opinion? For at present science itself seems compelled to suspend judgment.

As regards the astronomical side, other theories of the expanding universe are in the field. That of E. A. Milne has been more fully expounded since this book was written, and it offers a totally different approach to the subject. How far it will be found to give a fundamentally different explanation remains to be seen. But even general relativity offers other alternatives, as for example the Einstein-de Sitter solution of 1932, which dispenses with the cosmical constant altogether. Besides this we recall that where it has so far been possible to test the predictions of general relativity is in small corrections to Newtonian mechanics. When therefore we apply the theory to the whole universe and the "correcting" terms become dominant it behoves us to proceed warily lest these terms should turn out to be only the first of a series. Considerations of this nature tend to make us chary of basing so much on the one particular model of the universe chosen by Eddington. The recent book *Kosmos* by de Sitter contains some cautions in this direction.

As regards the quantum theory part of the subject, Professor Eddington is stating a development that is entirely his own. In consideration of the extent of the scientific issues involved we might venture with all deference to suggest the advantage of waiting for the views of other workers before setting it forth in popular form as an integral part of modern physical theory.

Eddington has recourse to Bottom the weaver. An apposite reply comes from the prologue to Bottom's play:

> "Gentles, perchance you wonder at this show;
> But wonder on, till truth makes all things plain".

W. H. McCrea.

**Ebene Geometrie.** By Albert Gminder. Pp. xvi, 490. RM. 22. 1932. (R. Oldenburg, Munich and Berlin)

The author's intentions are clear and reasonable.

He rejects Euclid's definitions in favour of new ones for which he claims certain advantages. He claims, among other things, that he does not use a construction before it is proved and that he does not rely on that method of proof (although it is a thoroughly valid one) in which a false assumption leads to a contradiction. That his intentions are clear and reasonable is some evidence that the author is, to a certain extent, successful; had he been completely successful we should, in a comparison with Euclid's presentation,

expect to find either a substantial shortening of the sequence or a deep insight into the nature of geometric concepts.

Euclid is dominated by two main ideas: the idea of sameness of size and the idea of sameness of direction. The author begins by defining the right angle by folding a sheet of paper. He folds it once and takes the crease as a straight line. He folds it again so that the two parts of the first crease coincide. When the sheet is unfolded the result displays what the author calls a right angle. So that the author, unlike Euclid, admits "folding" into his scheme. Like Euclid, however, he allows himself afterwards to be dominated by the concept of right angle. He defines area in terms of the square, while he defines sameness of direction in terms of two lines each perpendicular to a third line, these definitions being open to the criticism that neither of these concepts depends on that of right angle. All of this occurs in the first part of the book where the treatment is mainly experimental.

In the second part, the author makes everything depend on the circle, a knowledge of which has been got by experiment, by folding, etc. Here he defines parallel lines as lines which cut off equal arcs on a circle while angles are defined as central angles of a circle. He then shows that parallel lines are equidistant and proceeds by generalisation to define parallel curves. He says that $C$ and $C'$ are parallel curves if $PP'$ is constant, where $PP'$ is the normal at $P'$ on $C'$. Why authors take this as a definition of parallel curves it is difficult to see, for if $C'$ is an ellipse, $C$ could be a 4-cusped curve. It can be shown* that $P'P$ is also the normal at $P$ on $C$, and it is presumably on account of this property of equidistance that $C$ and $C'$ are said to be parallel. Later on, the author defines parallel motion by means of points $P$ and $P'$ which move on curves $S$ and $S'$ with the restriction $OP/OP'=$constant, so that if $S$ is an ellipse and $O$ its centre, $S'$ is also a similar ellipse and we have the result that parallel motion does not produce parallel curves!

In the second part of the book the author also treats elementary trigonometry from the geometric point of view. He finds all the properties of the *sinusstrecken* (what we in this country call sides opposite) and *cosinusstrecken*, and relies mainly on this for his development. He deduces the sum-formula from Ptolemy's theorem and discusses the historic relation, chord/arc at length. It is, however, doubtful whether it is advisable to make the whole of trigonometry thus depend on geometry. Nowadays, one is not so much concerned with what trigonometry once *was* as with what it now *is*. So far as the numerical part of geometry is concerned, a table of values of chord/arc is all that we need in addition to the theorems of Euclid and this is only necessary because of our bias to the right angle in computational work. But its application in geometry is only *one* of the many possible interpretations of trigonometry.

We can, for example, define $\sin x$ as the series $x-x^3/3!+x^5/5!\ldots$ where $x$ is a matrix, and all of the usual operational formulae remain valid. We mention this to show to what extent it is worth while to follow historic lines in the development of trigonometry.

The book is very well printed and there is at the beginning a list of abbreviations used and a good index at the end. So far as matter is concerned, it deals with the usual course in Euclid, together with work on such circles as the Tucker circle, harmonic section, pole and polar, antiparallels, sections of cones, etc. But we feel that the book could be much reduced in size without impairing the claims which the author makes. As it deals with material with which he is familiar, the teacher of mathematics who is learning German might use this book as a reader. He will also find many useful hints in the book: for example, the usual figures for proving the expansions for $\sin(a+b)$ and $\sin(a-b)$ are combined into one and (but the author does not note this fact) the result is the usual figure for $\sin P+\sin Q$. V. N.

* D'Ocagne, *Cours de Géométrie*, Tome 1, p. 129.

**Kurventheorie.** By K. MENGER. Pp. vi, 376. Geh. RM. 22. Geb. RM. 24. 1932. Mengentheoretische Geometrie in Einzeldarstellungen, 2. (Teubner)

The problem of defining a curve affords an excellent introduction to the theory of point-sets, and the book under review contains a thorough and attractive treatment of the problem. In the first chapter, called " Die alten Kurvenbegriffe ", it is handled by the methods initiated by Cantor. Typical of the chapter is an account of the most general set which can be the single-valued, continuous image of a linear segment. The second chapter introduces the more modern point of view, a curve being defined in terms of its intersection with an arbitrarily small neighbourhood of a general point. The theory is then developed in a way which strikes the reader as natural and, in the rather sophisticated way peculiar to the theory of sets, intuitive. For example, an idea which receives great emphasis is that of the order of a point, this being, roughly speaking, the number of arcs containing the point. The last three chapters contain accounts of trees, cyclic curves and universal curves. J. H. C. W.

**Principles of Descriptive Geometry.** By E. L. INCE. Pp. viii, 152. 8s. 6d. 1933. (Arnold)

In this book of some 152 pages the author gives a lucid explanation of the principles of descriptive geometry, which is assisted by the use of 153 diagrams, to be reproduced by the student himself step by step on a large scale, as the author rightly advises in his preface.

The seven chapters finish with a discussion of the simpler polyhedra, and the author wisely refrains from dealing with curved surfaces in this volume, but hints at the publication in the future of a second volume to consider cones, cylinders, surfaces of revolution, and the quadric surfaces. Such a volume written on the lines of the present one should prove a useful addition to the existing literature on the subject.

Chapter I in the present work commences with the purpose of descriptive geometry, and discusses the various projections and cartesian coordinates in three dimensions.

Chapter II deals with the point and the straight line in biorthogonal projection and shadows, whilst in Chapter III the plane in the same projection is dealt with.

Chapter IV is devoted to problems on the straight line and plane. The reader is introduced to displacements and their application in Chapter V. Chapters VI and VII deal with polyhedra and the figured plan.

Exercises are given at the end of each chapter, and at the end of the book are 28 practical exercises.

The subject does not lend itself to indexing, but nevertheless a short index might have been added with advantage.

The book is well and clearly printed and, although there are a good number of existing treatises on the subject, it should meet with a good reception, especially as it deals with the subject in a practical manner, and it should enable the student to get a thorough grasp of it in the one sure way, as the author says, namely, on the drawing-board. M. K. RICE-OXLEY.

**A First Trigonometry.** By S. N. FORREST. Pp. 96. 1s. 9d. 1933. (Arnold)

This little book purports " to set forth the essentials of Trigonometry up to the solution of triangles in a manner suited to the average pupil of a secondary school ", but if such a pupil can use projection on page 24 it seems a pity that he should not use projection at once and avoid the necessity of further explanations of the ratios of obtuse angles and still further work on the ratios of any angles. Teachers may find some of the examples useful, and some may like a means of getting the tangent formula for solving a triangle without the addition formulae. L. A.

**Mathematik und Bildende Kunst.** By WALTHER LIETZMANN. Pp. 150. Paper covers, RM. 4.00; Bound, RM. 5.50. 1931. (Hirt)

Of all the activities of man, Mathematics and Art are the two in which the idea of form most predominates. And in visual art, man thinks geometrically. The forms of architecture and of ornament are built up, sometimes obviously, sometimes so subtly as to conceal the framework in obscurity or deep subconsciousness, on the simple figures of the geometry books. The triangle, the square, the rectangle and the circle must always be at the back of men's minds. They are simple, pleasing, satisfying, and therefore beautiful. They crop up as soon as pencil meets paper; they stand before us in brick and stone all over the civilised world; and mankind is only too ready to attribute to them not only these virtues they possess, but a symbolical significance which is not inherently theirs.

It would be interesting to know more on this last point, in counterpart to the better-known mystical ideas of the Greeks and others about numbers, but Dr. Lietzmann has space for only one example, the curious "Yantra" figure from India, a combination of elements representing man and woman.

This is not, in the main, a deeply philosophical book, but simply a consideration of the way in which geometrical form enters into all kinds of visual art. As such, it might well have been made rather longer. After a brief six pages on the architecture of the Egyptians, to whom the author attributes considerable mathematical skill, we are given about as much on the Greeks and Romans, before being hurried on to an interesting comparison between Romanesque and Gothic, and the development from the simple Gothic forms dictated by statical necessity to the ornament of the Baroque and Rococo periods. There is also a brief, but interesting, section on the influence of modern building materials. But surely Dr. Lietzmann must have been tempted to add a few pages on the contribution of the engineer to geometrical design, with some illustrations of bridges and sky-scrapers.

He insists that it is by consideration of geometrical form, not of the overlaid ornament, that we can best assign a building to its own period. He does consider ornament, however, in the second part of the book, where he deals in turn with line-ornaments, stars, surface- and space-ornaments, and the decoration both of common objects and mathematical instruments. He might perhaps have found room among his many interesting examples for the problem of coffering a dome. He makes a short excursion into æsthetics when he asserts that the mathematician's criterion for a beautiful curve is that its curvature should vary continuously.

He then turns to the problem of plane-representation of a three-dimensional scene, and traces in some detail the development of perspective from the primitive drawings of the Babylonians and Assyrians (and modern children) to the perfect perspective of the photograph. He also discusses picture-analysis, both as regards the proportions of human figures and faces, and the grouping of figures in a picture.

It is easy enough to point to whole subjects that might have been included. For example, there is practically no discussion of form in post-impressionist and cubist art. But the reader of this book will agree that in the course of 150 very stimulating and well-illustrated pages the author has touched on a remarkable range of subjects and has left him full of new ideas. E. H. L.

**Théorie mathématique de l'Assurance invalidité et de l'Assurance nuptialité. Définitions et relations fondementales.** By H. GALBRUN. Pp. 156. 40 fr. 1933. (Gauthier-Villars)

This work is the fourth section of the third volume of the great *Traité du Calcul des Probabilités et de ses applications* produced by Professor Borel with the collaboration of a number of well-known mathematicians. In considering

the nature of this work we must emphasise the first two words in the title, for it is a mathematical theory, giving a short outline of the problem involved in insurance against Invalidity, and the provision of marriage benefits. For the mathematician, whose desire is merely to have an outline of how the theory of probabilities might be used in this subject of great social importance, the work may be commended for its logical clearness and its suggestiveness, but it will probably be found rather disappointing to the numerous workers engaged in the practical part of the subject.

The method followed by the author is (to use a hydrodynamical analogy) Lagrangian rather than Eulerian; thus, he considers the various adventures of a single life, passing out of a class $A$ (healthy), into a class $E$ (unhealthy), and discusses the case of a life having $n$ illnesses between the age $x$ and the age $y$; he then divides the interval and says, let $\nu$ be an *integer* less than $n$. This is the mathematician thinking of the abstract case, and is not the way in which the subject has been built up by the workers, who do not think of the individual, but of a group having at a given time a certain known consistency. At some future time they reconsider the new state of the group, to find what the inflow and outflow has been. Their method is not based on the theory of probabilities, their $p_x$'s and $q_x$'s are numbers obtained by observation, and these quantities would have been obtained, by the present date, if there had been no science of probability. Their method being statistical, it is only *after* the formation of the tables that they find what a useful tool the theory of probabilities is, in enabling them to obtain the values for complex combinations of lives.

But Dr. Galbrun's $p$'s and $q$'s are functions; he is thinking, not of number, but of structure, and having stated the problem in the form of an integral equation of the Volterra type, he then considers the value at certain points. The workers, however, have little belief in there being any law of mortality, marriage, sickness, etc., or that if one could be found that it would be the same to-morrow; their aim is to find a temporary working formula with a small number of constants, rather as an aid to their calculations than as the expression of any law of nature. For this purpose they have been drawn to the Error function and its successive differentials, as being perhaps the most useful of all functions, and the important work of the biometric school founded by Karl Pearson is having very great influence on the form of presentation.

The subject is of great social importance, and involves so much work that actuaries have come to an agreement on a system of notation, which seems admirably suitable for their purpose. Dr. Galbrun, unfortunately for them, has ignored this, and has given a notation often used in France, but little elsewhere.

The science of probabilities is the playground of the mathematicians, and they have given an example of the truth of Horace's line: *Dulce est discipere in loco.* No other subject contains so many curiosities, but for all this it has turned out to be a most useful one. Its birthplace was France, to which country we owe the greatest number of the most readable books on the subject, such as those of Laplace, Lacroix, Bertrand, Poincaré and Levy; the most critical work, however, is by the Russians, and the names of Tschebycheff, Markoff, Bortkiewicz and Tschuproff will naturally occur to us; but the widest application has been in Austria, Germany, England and America, where we find the best tables available. We should have liked to see some account of the old French work of Hubbard as modified by Lafitte, and that of Wiegand, Behm, Kihm, Rieder, and Weber; and of the Manchester Unity Table used in England, and used (both directly, and in a modified form by Cammerer) in America. The short table of Bentzien's given at the end of the work is much too scanty. Dr. Galbrun does give some account of Kuttner's Table, altering his notation so as to bring it into form with his own, and he rather

criticises Kuttner's illustration by the drawing from urns, as being false because we do not know the contents of the urns. This is really an illustration of the difference mentioned by De Morgan between the man who first tries the case and then settles the witnesses and the man who first settles the witnesses and then tries the case. Dr. Galbrun first forms his problem and then settles the state of the urn, but Kuttner first settles the state of the urn, and then forms his problem; this latter has been the most fertile in results. Kuttner's urn illustration seems, however, well chosen, for he does know the state of the urn at certain intervals, and he is quite satisfied to ignore the hidden forces, which do not affect the results. This method of illustrating by drawing from urns is useful for popular consumption and is hoary with antiquity; in fact we see in the last book of the *Iliad* that Zeus is described as having on his floor two urns with evil things and one with good things. From the consideration of the drawings he seems to have made, we might perhaps conclude that he is not ambidextrous.

That this work leaves out so much that we might have hoped for is probably due to its being only a section of the encyclopaedic *Traité*, and this due lack of proportion is characteristic of all encyclopaedias. We may remember what De Morgan said about the *English Encyclopaedia*: "Mathematics" has only five columns, "Mathematics, recent terminology of" has eight.

While acknowledging our appreciation of the book as a mathematical contribution to the subject, we regret that so much practical work has been ignored, and that the author has not stooped to the less mathematically-minded workers and introduced the notation with which they are familiar, and, instead of speaking of the ages attained, had introduced the time ($t$) as a parameter.

W. S.

**Exercices d'Analyse.** III. Equations Différentielles. By G. JULIA. Pp. iv, 287. 60 fr. 1933. (Gauthier-Villars)

Professor Julia, well known as the author of one of the most charming of the monographs in the *Collection Borel*, has had the happy idea of providing what may best be described as a "companion" to Goursat's *Cours d'Analyse*. He has collected a number of examples, some from Goursat's book, others from various examinations at the Sorbonne, and used them as the material for an interesting and helpful manual; by comparing different solutions of the same problem, indicating the ideas on which they depend, suggesting generalisations and connections with other branches of mathematics, the author endeavours to ensure that the student will acquire facility in the use of the elegant methods supplied in the *Cours d'Analyse*. Since the first and second volumes have not been noticed in the *Gazette*, we give a passage from the general preface in which Professor Julia outlines his intentions very lucidly:

"Le nombre des exercices développés n'est cependant pas très élévé: cela tient à la méthode de travail choisie pour des conférences dans lesquelles on s'efforce de traiter à fond, en variant le plus possible les méthodes, un petit nombre d'exercices choisis. Cette méthode peut se justifier par diverses raisons:

1°. Les esprits de caractéristiques différentes n'empruntant pas les mêmes voies pour arriver à la solution d'un problème, il importe de varier les voies d'accès à la solution.

2°. On peut souvent rattacher une même question à plusieurs théories du cours et l'étude de cette question ne pourra que gagner au rapprochement de ces théories.

3°. C'est par la comparaison de diverses solutions d'une même question qu'on saisit le mieux les avantages de chacune.

4°. Une solution longue, mais naturelle, est plus intelligible et plus profitable à l'étudiant qu'une solution rapide et artificielle. Et il arrive souvent que, valable pour le problème posé, elle ouvre en outre des horizons sur des questions connexes."

The present volume deals with ordinary differential equations; there are three chapters concerned respectively with elementary methods of integration, linear equations and singular solutions. Since this is not a systematic treatise, the author very rightly passes over many important topics in order to concentrate attention on those points where students are most likely to get into difficulties. Moreover, he does not neglect methods and illustrations from the theory of functions and from geometry, so that the student should draw from this volume not only facility in manipulating differential equations but also an appreciation of the profound and fundamental connections which entitle the theory of differential equations to be regarded as the central point of pure mathematics.

Many points of detail are worthy of notice; we have only room to remark how, in Professor Julia's skilful hands, the method of variation of parameters, so often regarded as merely an elegant futility, becomes a powerful and efficient auxiliary.

We are glad to recommend Professor Julia's book to all serious students of the subject; well written and full of interest, it merits a place on the shelf next to Goursat. T. A. A. B.

**Höhere Mathematik für Mathematiker, Physiker und Ingenieure.** Teil IV, Heft 1. By R. Rothe. Pp. ii, 52. RM. 2. 1932. Teubners Mathematische Leitfäden, 33. (Teubner)

The first and second volumes of Dr. Rothe's valuable text-book have already been noticed in the *Gazette* (XV, 123, 177). The third volume has not yet been published. The fourth volume is to be issued in six parts, two corresponding to each of the three main volumes; of each pair, one part is to contain examples and solutions and the other a collection of formulae. We have here the first part, consisting therefore of examples appropriate to the first volume, and so dealing with the foundations and elementary parts of the Calculus.

If English books on the calculus sometimes are overloaded with examples, Continental books seldom contain a sufficient number of genuine examples, where by genuine examples we mean those which are neither trivial corollaries to the book-work, nor well-known results for which room cannot be found in the text. But Dr. Rothe obviously believes that appreciation of theory and manipulative technique can and should be developed together and his selection of examples is intended to assist this dual development. None of the examples should cause any difficulty to, say, a good Scholarship candidate, so that the collection is essentially elementary, yet the range is wide and well-covered. There is a supply of functions for differentiation and integration, but they are not allowed to crowd out questions about curves such as $y=x^2\sin(1/x)$ and $y=x^3\sin(1/x)$, which demand a sound knowledge of continuity and differentiability, nor are practical applications excluded. Answers, hints and notes are given; they are conveniently but dangerously placed immediately after the question to which they belong.

We hope that Dr. Rothe will be able to publish the third volume and the remaining parts of the fourth in the near future and so round off a sound, instructive and eminently useful manual. T. A. A. B.

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